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MATHEMATICAL QUESTIONS,

WITH THEIR

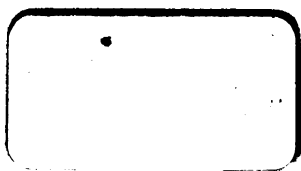
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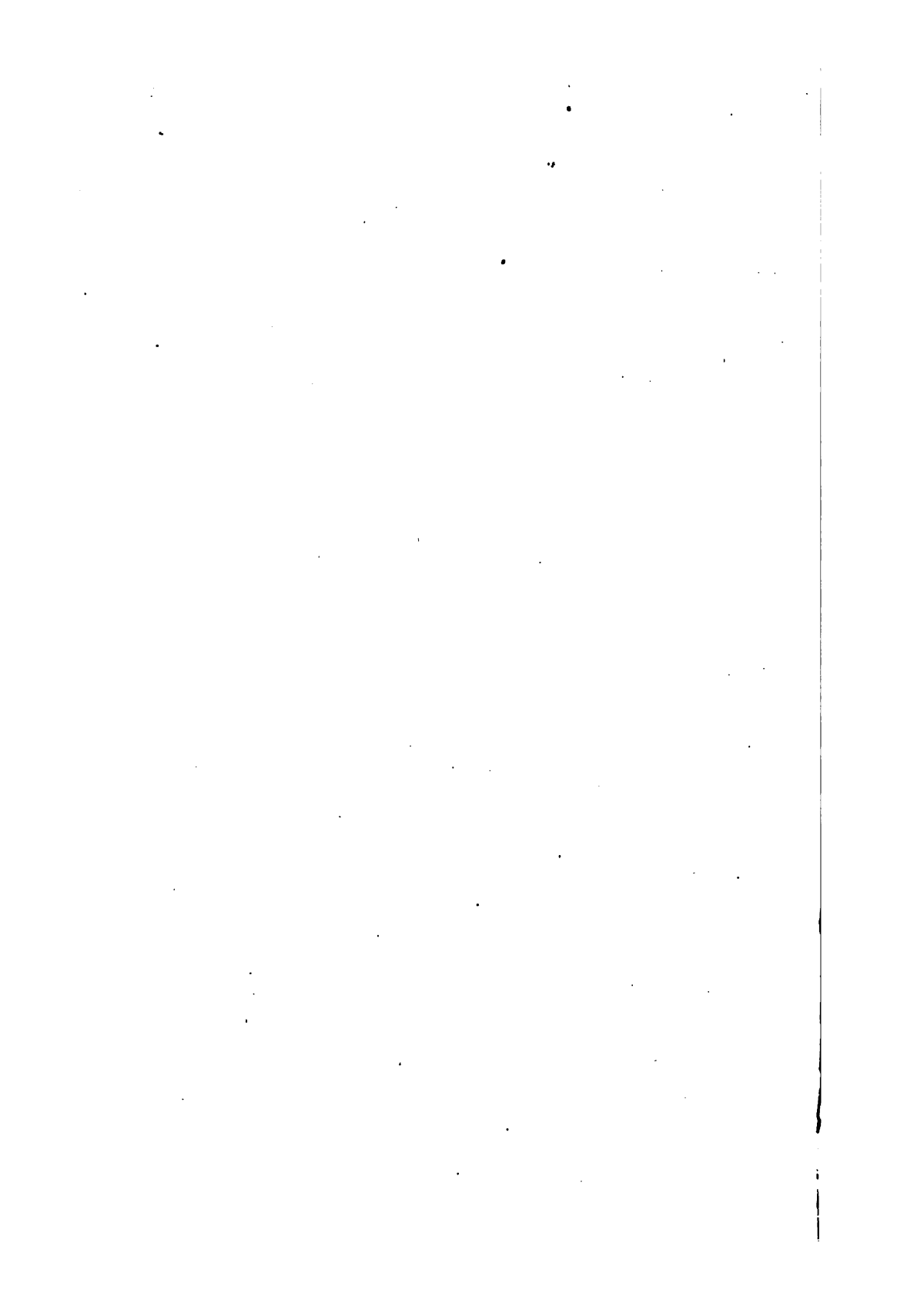
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MATHEMATICAL QUESTIONS,

WITH THEIR

SOLUTIONS,

FROM THE "EDUCATIONAL TIMES,"

WITH MANY

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CONTENTS.

Mathematical Papers, &c.

No.	Description of an Instrument for Drawing Quartic Binodal Curves. By S. Roberts, M.A.	Page.
89	Proof of Pascal's Theorem. By F. D. Thomson, M.A.	27
90		83

Solved Questions.

1058. (The Editor.)—A straight piece of wire is bent at random into two arms, and suspended by an extremity. Show that the chance that the angle will, in the position of equilibrium, rise above the point of suspension, is $\frac{1}{2} - \frac{\sqrt{2}}{\pi} \log(\sqrt{2} + 1)$, that is, $\cdot 10325$, or $\frac{1}{16}$ nearly 102
1059. (The Editor.)—The two straight arms of a bent lever are of the lengths a, ka (where $k > 1$), and contain an obtuse angle the cosine of whose supplement is λ^2 ; show that if from the ends of each of the arms a piece be cut at random, and the lever be then placed on a fulcrum at the angle, the probability that, in the position of equilibrium, one of the arms will rise above the fulcrum, is $\frac{\lambda(1+k^2)}{2k}$ or $\frac{\lambda^2+2\lambda k-1}{2\lambda k}$, according as the shorter arm does or does not rest above the fulcrum when the lever is suspended there before the cutting 100
1264. (Geometricus.)—On donne un angle ABC, un point M et une longueur a , et on demande de faire passer par le point M une droite telle que la partie XY interceptée entre les côtés de l'angle soit égale à la droite donnée a 103
1279. (The Editor.)—From a point O in the plane of a triangle ABC, of which the area is Δ , the centre of the circumscribed circle P, and its radius R, draw perpendiculars OD, OE, OF to the sides BC, CA, AB; and let Δ_1 be the area of the triangle DEF, and

No.	Page
OP = k ; then prove (1) that $\Delta_1 = \frac{1}{4} \left(1 - \frac{k^2}{R}\right) \Delta$; and hence show (2) that when Δ_1 is constant the locus of O is a circle with centre P and radius k ; (3) that the maximum value of Δ_1 is $\frac{1}{4}\Delta$, when O coincides with P; and (4) that Δ_1 degenerates into a straight line when O is on the circumscribed circle	95
1281. (The Editor.)—From the middle points of the sides of a triangle draw perpendiculars, proportional in length to these sides, and join the ends of the perpendiculars with the opposite corners of the triangle; then prove (1) that the joining lines will meet in a point, and (2) that, for varying lengths of perpendiculars, the locus of this point is the circumscribed rectangular hyperbola, whose trilinear equation is	
$\frac{\sin(B-C)}{\alpha} + \frac{\sin(C-A)}{\beta} + \frac{\sin(A-B)}{\gamma} = 0$	111
1321. (The Editor.)—If two marbles are thrown at random on the floor of a rectangular room, whose length is a and breadth b , show (1) that the probability of their resting at a distance apart less than r is $\frac{r^2}{ab} \left\{ \pi - \frac{4}{3} \left(\frac{r}{a} + \frac{r}{b} \right) + \frac{1}{2} \frac{r^2}{ab} \right\}$, when r does not exceed b ; (2) extend the problem to the case when r is greater than b ; and (3) show therefrom that for a square of side a , the respective probabilities when r has the values $\frac{1}{4}a$, $\frac{1}{2}a$, $\frac{3}{4}a$, are $\frac{1}{16}(\pi - \frac{5}{4}) = .1566$, $\frac{1}{4}(\pi - \frac{3}{2}) = .4833$, $\pi - \frac{5}{8} = .9583$, .9986 ...	90
2104. (William Curtis Otter, F.R.A.S.)—Required a full investigation of the following astronomical problem:—(1) by properties of the cone, (2) independently of the cone. To find the locus of the extremity of the shadow of a vertical gnomon, erected on a horizontal plane, on a given day, in a given latitude	97
2107. (N'Importe.)—A certain territory is bounded by two meridian circles, and by two parallels of latitude, which differ in longitude and latitude respectively by one degree, and is known to lie between the limits α and β of latitude; show that, if r be the radius of the earth, the mean superficial area is	
$\frac{\pi r^2 \sin \frac{1}{2} \cos \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\beta - \alpha - 1)}{45 (\beta - \alpha - 1)}$	99
2109. (N'Importe.)—A regular hexagon is composed of six equal heavy rods each of weight w moveable about their angular points, and two opposite angles are connected by a horizontal string; one rod is placed on a horizontal plane, and a weight W is placed at the middle point of the highest rod; show that the tension of the string is $\frac{1}{2} \sqrt{3} (W + 3w)$	101
2379. (S. Watson.)—Find the mean distance of all the points in a rectangular parallelepiped from one of its angles; and hence show that the mean distance in a cube whose side is unity is	
$\frac{3^{\frac{1}{2}}}{4} - \frac{\pi}{24} + \log \frac{1+3^{\frac{1}{2}}}{2^{\frac{1}{2}}}$	39

No.	Page
2671. (H. S. Hall).—If p_n denotes the ratio of the product of the first n odd numbers to the product of the first n even numbers, prove that	
(1)..... $p_{2m} - p_1 p_{2m-1} + p_2 p_{2m-2} - \dots + (-1)^{m-1} p_{m-1} p_{m+1}$ $= \frac{1}{2} p_m \{ 1 - (-1)^m p_m \},$	
(2)..... $\frac{p_{2m}}{4m-1} + \frac{p_1}{1} \cdot \frac{p_{2m-1}}{4m-3} - \frac{p_2}{3} \cdot \frac{p_{2m-2}}{4m-5} + \dots + (-1)^m \frac{p_{m-1}}{2m-3} \cdot \frac{p_{m-1}}{2m+1}$ $- \frac{1}{2m-1} \{ 1 + (-1)^m \frac{p_m}{2m-1} \}.$ <p style="text-align: right;">..... 77</p>	77
2970. (Professor Crofton, F.R.S.).—An ellipse, one of whose foci F is fixed, has double contact with a fixed ellipse E: show that its other focus describes an ellipse confocal with E and passing through F.	70
3123. (A. Martin).—Find the mean distance of the vertex of a right pyramid whose base is a rectangle (1) from all the points in its base, and (2) from all the points within the pyramid.	39
3312. (Dr. J. Matteson).—It is required to find four square numbers such that (1) the sum of their roots shall be a square, (2) the sum of the first and fourth, (3) the sum of the second and fourth, (4) the sum of the roots of the third and fourth multiplied by the root of the third, or (5) by the root of the sum of the second and fourth, (6) the sum of the first three added to three times the third, and (7) the sum of all four added to three times the third,—shall all be squares; (8) the product of the third and fourth added to the product of the first and second, (9) to the product of the first and third, (10) to the product of the second and third, or (11) to the sum of these three products,—shall produce squares; (12) any one of the numbers added to the sum of the roots of the other three, (13) the square of the sum of any two of their roots added to the sum of the other two roots, or (14) the square of the sum of any three of their roots added to the remaining root,—a square; (15) also, four other square numbers such that either of the first three added to the sum of the roots of the other three shall all be squares, and which also shall fulfil conditions (2, 3, 4, 5, 6, 7, 8, 9, 10, 11).....	35
3380. (Professor Townsend, F.R.S.).—Assuming that one focus of the shadow of a sphere, standing on a horizontal plane exposed to the light of the Sun, is its point of contact with the plane; find the envelope of the corresponding directrix, and the locus of the remaining focus, for a given day and place.....	52
3405. (Professor Wolstenholme).—If S, H be two fixed points at a distance $2a$, C the middle of SH, P a point such that the rectangle SP . HP = c^2 ($< a^2$); show that the radius of curvature of the locus of P at any point P is $\frac{b^2 c^2 \sqrt{2}}{a^3 - b^3}$, where $a^2 \equiv a^2 \cos 2\theta + b^2$, $b^2 \equiv a^2 \cos 2\theta - b^2$, $b^2 \equiv (a^4 - c^4)^{\frac{1}{2}}$, and $\theta = \angle PCS$. Hence prove that the locus of the points of minimum curvature, for different values of c , is the lemniscate $r^2 = a^2 \cos 2\theta$; and that the tangent at any point of minimum curvature passes through C. ...	42

No.		Page
3436.	(J. J. Walker, M.A.)—Tangents are drawn from a point (x', y') to the semi-cubic parabola $x^3 - 3ay^2 = 0$: show that the circle passing through the points of contact has for its equation $ax'(x^2 + y^2) - 3x'^2(x' + a)x + 2ay'(2x' + a)y + 4a(x' + a)y^2 = 0$; also determine and trace the locus of (x', y') when the circle touches the parabola at a point which is <i>not</i> one of the points of contact of the tangents from (x', y')	29
3461.	(J. W. L. Glaisher, B.A.)—Prove that $\int_0^\infty \left\{ e^{-qx} \text{Ei } qx - e^{qx} \text{Ei } (-qx) \right\} \sin px \, dx = \frac{q\pi}{p^2 + q^2}$ $\int_0^\infty \left\{ e^{-qx} \text{Ei } qx + e^{qx} \text{Ei } (-qx) \right\} \cos px \, dx = -\frac{q\pi}{p^2 + q^2},$ in which the function Ei (the exponential-integral) is defined by the equation $\text{Ei } x = \int_{-\infty}^x \frac{e^x}{x} \, dx$	22
3490.	(R. Tucker, M.A.)—B', C', D' are the centres of the osculating circles at the points B, C, D of an ellipse, which co-intersect at a point A on the curve, and O is the centre of the circle through A, B, C, D. Prove (1) that B', C', D' have precisely the same relations on an ellipse, centre O, with axes parallel to the original axes, as B, C, D have on the given ellipse; and (2) that the locus of the centroid of the five points A', B', C', D', O is a sextic curve.	38
3521.	(S. Watson.)—Within a given square a point is taken at random, and also upon the sides four points are taken at random, one on each; show that it is an even chance that the first point will lie within the quadrilateral formed by joining the other four	21
3531.	(The Rev. W. A. Whitworth, M.A.)—If a particle describe an ellipse under the action of a force tending to any fixed point O, prove that the law of force is $F \propto \frac{DD'^6}{OP \cdot PP'^3}$, where P is the position of the particle, PP' the chord through O, and DD' the diameter parallel to this chord.	108
3575.	(G. S. Carr.)—In a toy called Chinese Rings, a number of rings are hung upon a bar in such a manner that the ring at one end (A) can be put on or off the bar at pleasure; but any other ring can only be put on or off when the next one to it towards (A) is on, and all the rest towards (A) off the bar. The order of the rings cannot be changed. Show that if there are sixty rings, it will be necessary, in order to disconnect them from the bar, to put a ring either on or off 768614336404664650 times.	31
3580.	(T. Mitcheson, B.A.)—If the magnifying power of the common Astronomical Telescope, whose theoretic magnifying power is μ , becomes μ_1, μ_2 , &c., for ν persons whose distances of distinct	

CONTENTS.

ix

No.		Page
	vision are δ_1, δ_2 , &c., respectively, and if ϕ = the focal length of the eye-glass, show that	
	$\phi = \frac{\delta_1 (\mu_1 - \mu) + \delta_2 (\mu_2 - \mu) + \&c. + \delta_r (\mu_r - \mu)}{\mu \nu}$	80
3591.	(Professor Cayley.)—If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation $\alpha \cdot \text{PAB} + \beta \cdot \text{PBC} + \gamma \cdot \text{PCD} + \delta \cdot \text{PDA} = 0$, which connects the areas of the triangles PAB, &c.	61
3602.	(G. S. Carr.)—If a particle starting from rest at the point (a, b) be attracted to the axes of X and Y by forces $-\nu y, -\mu x$ respectively, show that the equation to the path is $\nu^{\frac{1}{2}} \cos^{-1} \frac{x}{a} = \mu^{\frac{1}{2}} \cos^{-1} \frac{y}{b},$ which includes all the curves made by Mr. H. AIRY's pendulum, and described by him in the July and September numbers of <i>Nature</i>	56
3604.	(The Editor.)—Find x and y from the equations $x + 2y - xy^2 + a(1 - 2xy - y^2) = y + 2x - x^2y + b(1 - 2xy - x^2) = 0$	81
3624.	(C. W. Merrifield, F.R.S.)—Show that the locus of the centres of all quadrics, intersecting in a curve, is a twisted cubic, such that from every point of it, as a vertex, a quadric cone can be drawn, the surface of which shall contain the cubic; and, moreover, that no other twisted cubic fulfils this condition.	17
3635.	(S. Watson.)—Find the average length of the radius of curvature in an ellipse	101
3636.	(Rev. W. H. Lavery, M.A.)—P is a point on an ellipse of which the foci are S and S'. PS, PS' meet the curve again in Q and Q'. RQM is perpendicular to the major axis, RM being equal to the central abscissa of Q'. Show that the locus of R is a rectangular hyperbola.	49
3638.	(R. W. Genese, B.A.)—Prove geometrically that the straight line joining the middle points of BC and AD, (D being the point where the inscribed circle touches BC,) passes through the centre of the inscribed circle	94
3688.	(G. M. Minchin, M.A.)—A particle moves from rest under the attraction (according to the inverse square) of a uniform straight rod. Show that it will continue to describe the hyperbola whose foci are the extremities of the rod, if it be acted on, in addition, by a repulsive force from the centre of the rod equal to $\mu r + \mu' r \log \left(\frac{(r^2 + b^2)^{\frac{1}{2}} - x}{(r^2 + b^2)^{\frac{1}{2}} + a} \right)^{\frac{1}{a}} \cdot \left(\frac{(r^2 + b^2)^{\frac{1}{2}} + c}{(r^2 + b^2)^{\frac{1}{2}} - c} \right)^{\frac{1}{c}},$ a and b being the semi-axes of the hyperbola, and $2c$ the length of the rod.	23
3691.	(Dr. Hart.)—In a plane triangle, find integers for the sides, diameter of the inscribed circle, and the side of the inscribed square, such that the sum of the diameter and side shall be a square.	48

No.		Page
3693.	(S. Watson.)—A line is drawn at random so as to cut a given triangle, and two points are taken at random within the triangle. Find the probability that the two points lie on opposite sides of the line; and hence show that, if the triangle be equilateral, the probability becomes $\frac{1}{3} + \frac{1}{15} \log 3$.	47
3696.	(G. S. Carr.)—A uniform rod of length a , elastic only to flexure, has its ends fastened to a string of length $b < a$. Determine the nature of the curve	66
3697.	(C. Taylor, M.A.)—If O be the orthocentre of a triangle ABC , prove that the straight lines joining the mid-points of AO , BC ; BO , CA ; CO , AB make with AB , BC , CA angles complementary to C , A , B .	65
3716.	(S. Tebay, B.A.)—A small marble is thrown at random on a square table, having an elevated rim. If it be struck at random in any direction, determine the probability that it impinges (1) on two opposite sides; (2) on two adjacent sides, and one opposite; (3) on three consecutive sides; (4) on the four sides in succession.	24
3718.	(F. D. Thomson, M.A.)—The locus of a point such that a tangent from it to a given conic is cut in a given ratio by a given straight line is a curve whose equation is of the form $pR \cdot UV + qRJ\mathfrak{Z} + rJ^2LM = 0,$ where p, q, r are constants; J is the line at infinity; U, V are the asymptotes of the given conic; L, M the tangents at the points where the given line R meets the conic; and \mathfrak{Z} is a certain conic having asymptotes parallel to U and V . Trace the possible changes of form which the locus may assume.	44
3724.	(M. Collins, B.A.)—If N be a given number, x^3 the nearest integral cube number $= N + D$; prove that $N^{\frac{1}{3}}$, which is $= x$ nearly, and $= x + d$ exactly, $= \frac{x(7N^3 + 105N^2x^3 + 120Nx^6 + 11x^9) + 21d^6x^3(N - x^3) - d^7(10x^3 - N)}{N^3 + 60N^2x^3 + 147Nx^6 + 35x^9}$ exactly $= \frac{(243N^3 + 378N^2D + 153ND^2 + 11D^3)x - 21Dx^2d^6}{243N^3 + 459N^2D + 252ND^2 + 35D^3}$ very nearly $= \frac{3N(9x^3 - 2D)^2 + D^2(6x^3 + 5D)}{3N(18x^3 - D)^2 + 141D^2x^3 - D^3} \times 4x$ nearly.....	34
3725.	(C. Taylor, M.A.)—The bisectors of the vertical angle A of a triangle meet the base in P, Q ; trace the variations of magnitude in $AO : OP$ and $AO : OQ$ as O moves along the base, and apply the result to prove that a conic is concave to its axis.....	21
3735.	(Artemas Martin.)—A ball is fired at random from a gun, with a given velocity; find the average of its range.	61
3741.	(Professor Townsend, F.R.S.)—Show that, when the curve of intersection of two quadric surfaces lies on a sphere, both surfaces have the same two systems of cyclic planes; and, conversely, that, when two quadric surfaces have the same two systems of cyclic planes, their curve of intersection lies on a sphere.	73

No.		Page
3744.	(Rev. H. T. Sharpe, M.A.)—To a smooth horizontal plane is fastened a hoop of radius c , which is rough inside, μ being the co-efficient of friction. In contact with this, a disc of radius a is spun with initial angular velocity n , and its centre is projected with velocity v in such a direction as to be most retarded by friction. Show that after a time $\frac{c-a}{\mu} \frac{k^2}{v} \frac{an+v}{a^2v-ank^2}$, the disc will roll inside the hoop.....	33
3746.	(Professor Wolstenholme, M.A.)—A uniform chain is in equilibrium under the action of any forces; from a fixed point O is drawn OP' parallel to the tangent at any point P of the chain, and of length proportional to the tension at P. Prove (1) that the tangent at P' to the locus of P' is parallel to the resultant force at P, and (2) that the ultimate ratio of small corresponding arcs at P', P is proportional to the force at P.	107
3749.	(Professor Crofton, F.R.S.)—An inflexible horizontal bar is suspended by three equal and similar vertical rods, attached at its middle and ends; a given weight being attached by a hook to any given point of the lower surface of the bar, determine the tensions of the three rods; show that, for certain positions of the weight, one of the rods will be <i>compressed</i> (as may be verified by experiment).	28
3751.	(The Editor.)—Two points taken at random on the surface of a sphere are joined to each other and to a third point taken at random either within or on the surface; or again, the centre of a sphere is joined to two points taken at random, either one or both within or on the surface. Find, in each case, the probability that the triangle thus formed is acute; also find the analogous probabilities in a circle.	26
3756.	(M. Collins, B.A.)—If a cardioid whose cusp is C be described by a moving body attracted towards an interior point C', prove that this attraction towards C' varies as $\frac{PC \cdot PC'}{(CP \cdot C'P^2)}$; PP' touching the curve at P and meeting C'P' drawn through C' parallel to CP	70
3757.	(Miss Ladd.)—In a quadrilateral ABCD, join the centres of gravity of the triangles ABC, ACD, ABD, BCD, and show that the resulting quadrilateral will be similar to ABCD, and that each side will be equal to one third of the corresponding side of ABCD.	37
3758.	(J. J. Walker, M.A.)—Determine y as a function of x from $3 \left\{ \frac{d^2 y}{dx^2} \right\}^2 + \frac{d \left(\frac{u}{v} \right)^2}{dx} \cdot \frac{d^2 y}{dx^2} = 0,$ u and v being given functions of x only.	29
3760.	(G. S. Carr.)—Find when the light thrown upon a round table from a luminous point over the centre is a maximum; also when a smaller concentric circle is excluded from the surface...	44

No.		Page
3764.	(B. Williamson, M.A.)—Prove that the ordinary solution for the position when a planet appears brightest gives no real position if the planet be nearer to the sun than one-fourth of the earth's distance.	32
1766.	(A. Stein.)—Prove that for all real values of a_1, a_2, \dots, a_n , $\sum_{k=1}^{k=n} \frac{a_k^{n-2}}{(a_k - a) \dots (a_k - a_{k-1})(a_k - a_{k+1}) \dots (a_k - a_n)} = 0. \dots$	43
3766.	(R. Tucker, M.A.)—If ρ_1, ρ_2, ρ_3 be the inscribed radii of the triangles $D_1E_1F_1, D_2E_2F_2, D_3E_3F_3$, formed by joining the points of contact of each of the escribed circles with the three sides BC, CA, AB of a triangle ABC; prove that $\frac{1}{\rho_1} : \frac{1}{\rho_2} : \frac{1}{\rho_3} = 1 - \tan \frac{1}{4}A : 1 - \tan \frac{1}{4}B : 1 - \tan \frac{1}{4}C. \dots$	30
3770.	(T. Mitcheson, B.A., F.C.P.)—If x_1, px_1 be the abscissas of two points on a parabola, and x_3 the abscissa of the point of intersection of the tangents at the first two points; prove that $x_3 = p^{\frac{1}{2}}x_1. \dots$	40
3773.	(Rev. Dr. Booth, F.R.S.)—1. Let $Ax^2 + A_1y^2 + 2Bxy + 2Cx + 2C_1y = 1$ be the projective equation of a conic, and let a right-angle move along this curve, one side passing through the origin; then the other side will envelope a curve whose tangential equation is $A\xi^2 + A_1v^2 + 2B\xi v + 2(C\xi + C_1v)(\xi^2 + v^2) = (\xi^2 + v^2)^2$. 2. Let $\alpha\xi^2 + \alpha_1v^2 + 2\beta\xi v + 2\gamma\xi + 2\gamma_1v = 1$ be the tangential equation of a conic, and from the origin let a perpendicular be drawn on a tangent to the conic; then the locus of the feet of the perpendiculars will be the curve, whose projective equation is $\alpha x^2 + \alpha_1y^2 + 2\beta xy + 2(\gamma x + \gamma_1y)(x^2 + y^2) = (x^2 + y^2)^2. \dots$	41
3774.	(Professor Townsend, F.R.S.)—Show that the two sheets of the biaxial wave surface, apsidal of the ellipsoid whose semi-axes are a, b, c , are reciprocal polars to each other with respect to the coaxial ellipsoid whose corresponding semi-axes are $(bc)^{\frac{1}{2}}, (ca)^{\frac{1}{2}}, (ab)^{\frac{1}{2}}. \dots$	79
3776.	(Professor Wolstenholme, M.A.)—1. Show that the "feet-perpendicular" lines of any triangle inscribed in a circle are all asymptotes of rectangular hyperbolas passing through the corners of the triangle. 2. If S be the focus of the parabola inscribed in a triangle ABC; Aa, Bb, Cc the perpendiculars of the triangle; S' a point on the nine-point circle such that S, S' are at the ends of parallel and opposite radii of the circles ABC, abc; show that the tangent at the vertex of the former parabola will be the axis of a parabola inscribed in abc, and having its focus at S'.	54
3778.	(T. Cotterill, M.A.)—1. If 1, 2, 3, 4, 5 are the successive sides of a plane pentagon, show that the tangent at the point (2, 4)	

CONTENTS.

xiii

No.	Page
	to the conic through the five intersections of its non-successive lines cuts the line 3, in the same point as the line through the points (1, 2) and (4, 5).
	2. Prove that a cubic through these five points, touching the conic at the point (2, 4) and passing through the point (1, 5) cuts the lines 2, 3, 4 again in collinear points, and that two such cubics intersect again in two points collinear with (1, 5)... 65
3780.	(The Editor.)—If a, b, c, d be the lengths of the sides of a quadrilateral whose diagonals are equal and perpendicular to each other, show that its area is $\frac{1}{2} \{ a^2 + c^2 + [4b^2d^2 - (a^2 - c^2)^2]^{\frac{1}{2}} \}$... 50
3782.	(Professor Crofton, F.R.S.)—Find the distribution of a weight placed on a table among the four legs of the table, supposing the slab inflexible. 57
3787.	(C. Taylor, M.A.)—Show (1) that an ellipse which has double contact with two fixed confocals has a fixed director circle; and (2) that a variable quadric which has double contact with three fixed confocal quadrics has a fixed director sphere 19
3788.	(J. J. Walker, M.A.)—At a point where the normal to the central conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axis, a line is drawn parallel to the tangent. Show that the reciprocal polar to the envelope of this line, with respect to the circle described on the line joining the foci as diameter, is $a^2x^2 + b^2y^2 = x^4$ 88
3791.	(T. Mitcheson, B.A.)—If D be the middle point of the base BC of a triangle ABC, and E the foot of the perpendicular from the vertical angle; prove that $2BC \cdot DE = AC^2 - AB^2$... 34
3794.	(R. Tucker, M.A.)—AB, AC are two given indefinite straight lines touching a given circle in E and F; a variable line BC is drawn touching the circle, and BF, CE are drawn intersecting in P; show that the locus of P is an ellipse also touching the given lines AB, AC in E and F. 76
3795.	(J. F. Moulton, M.A.)—Show that the equation $x^n + rx^{n-p} + s = 0$ will have two equal roots if $\left\{ \frac{r}{n} (p-n) \right\}^n = \left\{ \frac{s}{p} (n-p) \right\}^p$ 18
3800.	(W. Hogg.)—A particle falls towards a centre of force, of which the intensity varies inversely as the cube of the distance; to find the whole time of descent. 45
3801.	(R. W. Genese, B.A.)—ABCD is a parallelogram inscribed in a rectangular hyperbola. Prove (1) that AB, CD opposite sides subtend either equal or supplementary angles at any point P of the curve; (2) that the circles circumscribing PAB, PBC, PCD, PDA, are equal; and (3) that the product of the perpendiculars from P on each pair of opposite sides is the same. 32
3803.	(Rev. Dr. Booth, F.R.S.)—If $f(x, y) \equiv F = 0$ be the projective equation of a plane curve, show that the tangential equation of

No.		Page
	its evolute may be found by eliminating x and y from the equations	
	$\xi = \frac{dF}{dy} \div \left(\frac{dF}{dy} x - \frac{dF}{dx} y \right), \quad v = -\frac{dF}{dx} + \left(\frac{dF}{dy} x - \frac{dF}{dx} y \right), \quad f(x, y) = 0;$	
	and show therefrom that the tangential equation to the evolute of the ellipse is $a^2v^2 + b^2\xi^2 = (a^2 - b^2)^2 \xi^2 v^2$.	58
3808.	(C. Taylor, M.A.)—In the right circular cone, if C, C' be the centres of a pair of focal spheres, show that the sphere on CC' contains the auxiliary circle of the section.	87
3809.	(Dr. Ball.)—Show that the tension of the string of a prismatic bow is $\frac{\pi^2 K I}{L^2} \left(1 + \frac{\pi^2 D^2}{8 L^2} \right)$, where K is the coefficient of elasticity, I the moment of inertia of the section of the bow, L the length of the bow, and D the distance between the centre of the string and the centre of the bow. Also explain why the tension does not vanish when D = 0.	66
3812.	(The Editor.)—A person borrows £10000: how long will it take him to clear off the debt by paying off £100 at the end of the first year, £200 at the end of the second year, £300 at the end of the third, and so on, allowing 5 per cent. compound interest on the whole?	59
3815.	(J. J. Walker, M.A.)—Show that the locus of feet of perpendiculars let fall from points in a given diameter of a quadric surface on the polar planes of those points is a rectangular hyperbola.	41
3817.	(M. Collins, B.A.)—A body will describe a circle by a force $= \phi^2 r^{-1}$ directed towards its centre. Hence prove that if a circle be drawn round any other interior point C as centre of attraction, the force of attraction towards C must vary as PC + (the cube of the distance of P from the polar of C); P being any position of the point moving in the circumference; and show that this last property is true for all conic sections.	62
3820.	(N'Importe.)—From a given point D in the produced diameter BA of a given circle ACP, to draw a line DCH, cutting the circumference in C and H, so that, if BC, BH be joined, the area of the triangle BCH may be given or a maximum.	78
3829.	(Rev. Dr. Booth, F.R.S.)—A right angle revolves round the origin of coordinates of a curve whose tangential equation is $\phi(\xi, v) \equiv \Phi = 0$, and one side of the angle meets a tangent to the curve in its point of contact; show that the other side of the right angle meets this tangent in a point whose locus may be determined in projective coordinates by the three equations	
	$x = \frac{d\Phi}{dv} \div \left(\frac{d\Phi}{dv} \xi - \frac{d\Phi}{d\xi} v \right), \quad y = -\frac{d\Phi}{d\xi} + \left(\frac{d\Phi}{dv} \xi - \frac{d\Phi}{d\xi} v \right), \quad \phi(\xi, v) = 0;$	
	and show therefrom that the locus for the ellipse is	
	$a^2x^2 (a^2x^2 + b^2y^2) = (a^2 - b^2)^2 x^2 y^2.$	58

CONTENTS.

IV

No.		Page
3831.	(C. W. Merrifield, F.R.S.)—Show that all cones satisfy the differential equations $\frac{\alpha\gamma - \beta^2}{r} = \frac{\alpha\delta - \beta\gamma}{2s} = \frac{\beta\delta - \gamma^2}{t}, \text{ where } \alpha = \frac{d^2x}{dx^2}, \beta = \frac{d^2x}{dx^2 dy}, \text{ \&c.;}$ and that, when the vertex moves off to infinity, the numerators vanish simultaneously.	60
3838.	(Professor Crofton, F.R.S.)—If the two principal stresses at any point of a plane lamina be equal and of contrary signs, show that the resultant stresses on the three sides of any triangular element of the lamina at that point meet on the nine-point circle of the triangle.	43
3839.	(A. B. Evans, M.A.)—Prove geometrically the following relations of a plane triangle, $p_1 + p_2 + p_3 = R + r, \quad -p_1 + p_2 + p_3 = -R + r_1 \quad \dots\dots (1, 2),$ $p_1 - p_2 + p_3 = -R + r_2, \quad p_1 + p_2 - p_3 = -R + r_3 \quad \dots\dots (3, 4);$ where R denotes the radius of the circumscribed circle; r the radius of the inscribed circle; r_1, r_2, r_3 the radii of the escribed circles; p_1, p_2, p_3 the perpendiculars from the centre of the circumscribed circle on the sides	85
3841.	(W. Siverly.)—A bowl which is a segment of a hollow sphere whose external and internal radii are R and r, and depth r - c, rests on a horizontal plane, and has a weight attached to the outer edge of the rim, is one -n th the weight of the bowl. Show that in its position of equilibrium the inclination of the rim of the bowl to the horizon is $\tan^{-1} \frac{4(R^2 - c^2)^{\frac{1}{2}} \{ (2R + c)(R - c)^2 - (2r + c)(r - c)^2 \}}{3n \{ (R^2 - c^2)^2 - (r^2 - c^2)^2 \} + 4c \{ (2R + c)(R - c)^2 - (2r + c)(r - c)^2 \}} \quad \dots\dots$	75
3842.	(T. Mitcheson, B.A.)—Two radii vectores, r and r ₁ , are drawn from the vertex of a parabola at right angles; show that the inclination of one of them to the axis of x is $\cot^{-1} r^{\frac{1}{2}} r_1^{-\frac{1}{2}}$	95
3845.	(C. Taylor, M.A.)—A triangle inscribed in an ellipse envelopes a confocal. Ellipses are described, each passing through a vertex, and having its foci at the adjacent points of contact. Show that the sum of the areas of their minor auxiliary circles is constant.	63
3846.	(W. H. H. Hudson, M.A.)—The <i>Times</i> of Thursday, Jan. 4, 1872, is numbered 27264. Supposing the paper to have been published every week-day without intermission, and numbered consecutively; give the day of the week, month, and year when the first number was published.	63
3849.	(R. Tucker, M.A.)—The circle through the focus of a parabola which touches the curve at P cuts it also in Q, R; it is known that QR passes through a fixed point (O) on the axis; find the locus of the intersection of the tangent at P with QR, and the area of the portion of it comprised between O and its asymptote. Show that the perpendicular from P on QR cuts it in a parabola, vertex O, and touches a semi-cubical parabola, whose vertex is the foot of the given directrix; also find the locus of the centre of the circle and the area of its oval. The centroid of PQR lies on the axis.....	55

No.		Page
3855.	(Professor Wolstenholme.)—Three points A, B, C move on the arc of a fixed circle so that the centre of perpendiculars of the triangle ABC is a fixed point; prove that their velocities are to one another as $\sin 2A : \sin 2B : \sin 2C$.	64
3856.	(The Editor.)—Three men bought a grinding-stone 48 inches in diameter, 8 inches thick at the centre, and 12 inches thick at the circumference; each paying one-third of the expense. What part of the radius must each grind down for his share?	69
3859.	(S. Watson.)—Two equal spheres intersect: show that the chance of the portion common to both exceeding half the volume of either, is $8 \cos^3 80^\circ$.	74
3860.	(W. H. H. Hudson, M.A.)—If $\frac{\cos(\beta + \alpha) + \cos(\alpha + \gamma)}{\cos(\beta - \alpha) + \cos(\alpha - \gamma)} = \frac{\cos(\beta + \gamma) + \cos(\alpha + \gamma)}{\cos(\gamma - \beta) + \cos(\alpha - \gamma)},$ prove that $\frac{\sin \beta}{\cos \gamma - \cos \alpha} + \frac{\cos \beta}{\sin \alpha - \sin \gamma} = \frac{1}{\sin(\alpha - \gamma)}$	85
3871.	(The Editor.)—In how many different ways may a bookshelf 4 feet 2 inches long be <i>exactly</i> filled with volumes $2\frac{1}{8}$, $2\frac{3}{8}$, $2\frac{5}{8}$ inches thick respectively, using at least <i>one</i> volume of each sort...	86
3872.	(W. C. Otter, F.R.A.S.)—Given a circle and a point not in the plane of the circle; find the centre of the sphere which passes through the given point and through the circumference of the given circle	75
3874.	(Sir James Cockle, F.R.S.)—Writing, for brevity, $\frac{dy}{dx} = p$, and $\frac{d^2y}{dx^2} = \frac{dp}{dx} = r$, let there be given $nx^3r = (y - xp)^2$(1). Now BOOLE (<i>Differential Equations</i> , 2nd ed., p. 218) in effect states that $y = cx$(2) is the singular solution of (1). Test this statement.	66
3875.	(Professor Cayley.)—Given the constant a and the variables x, y , to construct mechanically $\frac{a^2 - x^2}{y}$; or what is the same thing, given the fixed points A, B, and the moving point P, to mechanically connect therewith a point P' such that PP' shall be always at right angles to AB, and the point P' in the circle APB.	68
3883.	(M. Collins, B.A.)—Prove that the harmonic mean between the segments of any chord of a conic section passing through its focus is constant and equal to the semi-parameter.	84
3885.	(Professor Clifford.)—If A be the single focus of a semi-cubical parabola, there exists a straight line BC, such that if two tangents at right angles cut it in B, C, the angle BAC is also a right angle.	82
3887.	(F. D. Thomson, M.A.)—Show that the equation to the cone of the second order whose sides are the lines 1, 2, 3, 4, 5 terminating at the origin, may be written (014) (035) (245) (123) + (035) (013) (214) (245) + (025) (013) (435) (214) = 0, where (014) $\equiv \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \end{vmatrix}$ &c., (x_1, y_1, z_1) being a point in the line 1, &c.	71

No.		Page
3890.	(Professor Wolstenholme.)—Forces P, Q, R, P', Q', R' act along the edges BC, CA, AB, DA, DB, DC of a tetrahedron: prove that they will be equivalent to a couple only if $\frac{P'}{AD} = \frac{R}{AB} - \frac{Q}{CA}, \quad \frac{Q'}{BD} = \frac{P}{BC} - \frac{R}{AB}, \quad \text{and} \quad \frac{R'}{CD} = \frac{Q}{CA} - \frac{P}{BC} \dots$	87
3891.	(Dr. Hart.)—Find n numbers whose sum is a square, and the sum of their squares a biquadrate	104
3893.	(Dr. Ball.)—Prove (a) that the equation to the neutral axis of the bow is the result of eliminating ϕ between $x = \left(\frac{KI}{T}\right)^{\frac{1}{2}} \{2E(e, \phi) - F(e, \phi)\} \quad \text{and} \quad y = D(1 - \cos \phi), \quad \text{where } e^2 = \frac{D^2 T}{4KI}$ (b) That the entire length of the bow is $2 \left(\frac{KI}{T}\right)^{\frac{1}{2}} F(e)$. (c) That the length of the bow and string together is $4 \left(\frac{KI}{T}\right)^{\frac{1}{2}} E(e)$ 67	
3898.	(W. Siverly.)—A hollow cone made of thin paper, and having a vertical angle $= \alpha$, is cut open by a straight line from the vertex to a point in the base, and spread out so as to form a plane surface. Find the vertical angle of this surface	106
3900.	(W. H. H. Hudson, M.A.)—Six equal and similar unstretched light elastic strings are fastened together so as to form a square, and the two lines bisecting opposite sides. Four equal forces being applied at the angular points of the square along the diagonals outwards, obtain equations to determine the shape and size of the network	84
3904.	(Professor Wolstenholme.)—A string lies on a horizontal table in the form of any arc of a cycloid, not including a cusp, and the density at any point varies as the cube of the curvature; impulsive tensions are applied to its ends which are to one another as the respective curvatures: prove that the impulsive tension at any point will be proportional to the curvature, and that the whole string will move without change of form parallel to its axis.....	110
3906.	(Rev. Dr. Booth, F.R.S.)—The centres of a series of circles range along a parabola, their focal distances are f , and their radii are $= (f^2 - a^2)^{\frac{1}{2}}$, $4a$ being the parameter; find the curve which envelopes the circles	81
3908.	(Dr. Ball.)—When the effect of an impulsive force upon a free quiescent rigid body is to make the body revolve about an instantaneous axis, show that the force and the axis are parallel to the principal axes of a section of the momental ellipsoid about the centre of inertia	109
3911.	(C. H. Hinton.)—Show that $\frac{1}{2} n(2n-1)(n+5) + I(\frac{1}{2}n) = M(3)$. 99	
3924.	(B. Williamson, M.A.)—Prove that the eccentricity of any section of a right cone is represented by $\frac{\cos \beta}{\cos \alpha}$, where β is the angle made by the plane of section with the axis of the cone, and α is the semi-angle of the cone.....	88

No.		Page
3938.	(T. Dobson, B.A.)—If a straight line of a given length has one extremity in a vertical line, and the other in a horizontal plane, each assigned point of the line lies in the surface of a solid. Prove that the average volume of the solids whose surfaces are the loci of all the points in the given line is two-thirds of the volume of the solid whose surface is the locus of its middle point.....	105
3939.	(Dr. Hirst, F.R.S.)—Prove that if $F(x, y, a, b, c, f, g, h) = 0$ be the equation of the n th pedal of the conic $(a, b, c, f, g, h)(x, y, 1)^2 = 0,$ the equation of the $(-n+1)$ th pedal will be $F\left(-\frac{x}{r^2}, -\frac{y}{r^2}, \frac{d\Delta}{da}, \frac{d\Delta}{db}, \frac{d\Delta}{dc}, \frac{1}{2}\frac{d\Delta}{df}, \frac{1}{2}\frac{d\Delta}{dg}, \frac{1}{2}\frac{d\Delta}{dh}\right) = 0,$ where $r^2 = x^2 + y^2$, and $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$	106
3940.	(Dr. Salmon, F.R.S.)—Prove that the area of the triangle formed by the polars of the middle points of the sides of a triangle, with respect to any inscribed conic, is equal to the area of the given triangle.....	107
3952.	(T. Mitcheson, B.A.)—A is a point on the circumference of a given circle, P a point without it; AP cuts the circle in B, and $AP = n \cdot AB$; show that the locus of P is a circle touching a given circle in A.....	99

MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

3624. (Proposed by C.W. MERRIFIELD, F.R.S.)—Show that the locus of the centres of all quadrics, intersecting in a curve, is a twisted cubic, such that from every point of it, as a vertex, a quadric cone can be drawn, the surface of which shall contain the cubic; and, moreover, that no other twisted cubic fulfils this condition.

Solution by the PROPOSER.

1. Referring two quadrics of the system to axes parallel to conjugate diameters in each, and placing the origin at one of them, they may be written as

$$U \equiv 0 = a_1x^2 + b_1y^2 + c_1z^2 + d_1,$$

$$V \equiv 0 = a_2(x-a)^2 + b_2(y-\beta)^2 + c_2(z-\gamma)^2 + d_2;$$

and the centre of $U + kV = 0$ is given by $a_1x + a_2k(x-a) = 0$, &c.;

whence
$$-k = \frac{a_1x}{a_2(x-a)} = \frac{b_1y}{b_2(y-\beta)} = \frac{c_1z}{c_2(z-\gamma)} \dots\dots\dots (a),$$

which is a twisted cubic.

2. Next, let (p, q, r) be a point on this cubic; we have to prove that

$$\frac{a_1(x+p)}{a_2(x+p-a)} = \frac{b_1(y+q)}{b_2(y+q-\beta)} = \frac{c_1(z+r)}{c_2(z+r-\gamma)},$$

or, which is the same thing, since (p, q, r) is a point upon (a), that

$$\frac{a_1x}{a_2(\lambda x + p - a)} = \frac{b_1y}{b_2(\lambda y + q - \beta)} = \frac{c_1z}{c_2(\lambda z + r - \gamma)}$$

can be combined into a single homogeneous equation of the second degree. It is easily seen that this can always be done; for if we clear fractions in each of the three equations, multiply by x, y, z severally, and add, the cubic terms go out, and leave the equation of a cone.

3. Any twisted cubic may be regarded as the curved part of the inter-

section of two quadric cones having a common edge. Referring these to axes parallel to conjugate diameters in each, their equations are

$$a_1x^2 + b_1y^2 + c_1z^2 = 0, \quad a_2(x-\alpha)^2 + b_2(y-\beta)^2 + c_2(z-\gamma)^2 = 0 \dots (1, 2).$$

If these are to represent cones having as a common edge the line from the origin through (α, β, γ) , we must also have

$$a_1\alpha^2 + b_1\beta^2 + c_1\gamma^2 = 0, \quad a_2\alpha^2 + b_2\beta^2 + c_2\gamma^2 = 0 \dots (3, 4),$$

or

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\beta^2}{c_1a_2 - c_2a_1} = \frac{\gamma^2}{a_1b_2 - a_2b_1} \dots (5).$$

Let the point (p, q, r) satisfy (1) and (2), and let us transfer the origin to it. Now (1) and (2) must be written as

$$a_1(x+p)^2 + b_1(y+q)^2 + c_1(z+r)^2 = 0 \dots (6),$$

$$a_2(x+p-\alpha)^2 + b_2(y+q-\beta)^2 + c_2(z+r-\gamma)^2 = 0 \dots (7),$$

with the relations

$$a_1p^2 + b_1q^2 + c_1r^2 = 0,$$

$$a_2(p-\alpha)^2 + b_2(q-\beta)^2 + c_2(r-\gamma)^2 = 0,$$

which reduce (6) and (7) to

$$a_1x^2 + b_1y^2 + c_1z^2 + 2a_1px + 2b_1qy + 2c_1rz = 0 \dots (8),$$

$$a_2x^2 + b_2y^2 + c_2z^2 + 2a_2(p-\alpha)x + 2b_2(q-\beta)y + 2c_2(r-\gamma)z = 0 \dots (9).$$

No linear combination of these equations can be rendered homogeneous by any change of axes, unless

$$\frac{a_1p}{a_2(p-\alpha)} = \frac{b_1q}{b_2(q-\beta)} = \frac{c_1r}{c_2(r-\gamma)}.$$

But this is the locus of centres alluded to before. Hence that is the only twisted cubic which fulfils the condition.

3795. (Proposed by J. F. MOULTON, M.A.)—Show that the equation $x^n + rx^{n-p} + s = 0$ will have two equal roots if

$$\left\{ \frac{r}{n} (p-n) \right\}^n = \left\{ \frac{s}{p} (n-p) \right\}^p.$$

Solution by the Rev. R. HARLEY, F.R.S.; J. DALE, and others.

The given trinomial $\phi(x) = 0 = x^n + rx^{n-p} + s$ will have two equal roots if $\phi'(x) = 0$, that is, if

$$nx^{n-1} + (n-p)rx^{n-p-1} = 0, \quad \text{or if } x = \left\{ \frac{r}{n} (p-n) \right\}^{\frac{1}{p}},$$

and therefore if $\left\{ \frac{r}{n} (p-n) \right\}^{\frac{n}{p}} + r \left\{ \frac{r}{n} (p-n) \right\}^{\frac{n-p}{p}} + s = 0$,

that is, if $\left\{ \frac{r}{n} (p-n) \right\}^n = \left\{ \frac{s}{p} (n-p) \right\}^p$.

This may also be shown by writing

$$\phi(x) = x^p + sx^{p-n} + r = 0;$$

for then, by differentiation, we have

$$\phi'(x) = px^{p-1} + (p-n)sx^{p-n-1} = 0,$$

whence

$$x = \left\{ \frac{s}{p} (n-p) \right\}^{\frac{1}{p}},$$

and by equating this value of x with that obtained above, we have

$$\left\{ \frac{r}{n} (p-n) \right\}^{\frac{1}{p}} = \left\{ \frac{s}{p} (n-p) \right\}^{\frac{1}{n}},$$

in which either member is equal to one of the two equal roots.

3787. (Proposed by C. TAYLOR, M.A.)—Show (1) that an ellipse which has double contact with two fixed confocals has a fixed director circle; and (2) that a variable quadric which has double contact with three fixed confocal quadrics has a fixed director sphere.

Solution by Professor TOWNSEND, F.R.S.

1. The equations of the two fixed conics S_1 and S_2 being

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1, \quad \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = 1 \dots\dots\dots (1),$$

that of the variable conic S , which is evidently concentric with them, is

$$\text{consequently} \quad \left[\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - 1 \right] - \lambda_1^2 \left[\frac{x}{l_1} - \frac{y}{m_1} \right]^2 = 0 \dots\dots\dots (2),$$

where $x : y = l_1 : m_1$ is its diameter of double contact with S_1 ; or

$$\left[\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} - 1 \right] - \lambda_2^2 \left[\frac{x}{l_2} - \frac{y}{m_2} \right]^2 = 0 \dots\dots\dots (3),$$

where $x : y = l_2 : m_2$ is its diameter of double contact with S_2 . Hence, identifying coefficients of x^2 , xy , and y^2 in equations (2) and (3), and remembering that S_1 and S_2 are confocal, we get at once the four equations

$$\frac{l_1 m_1}{\lambda_1^2} = \frac{l_2 m_2}{\lambda_2^2}, \quad \frac{\lambda_1^2}{l_1^2} - \frac{1}{a_1^2} = \frac{\lambda_2^2}{l_2^2} - \frac{1}{a_2^2}, \quad \frac{\lambda_1^2}{m_1^2} - \frac{1}{b_1^2} = \frac{\lambda_2^2}{m_2^2} - \frac{1}{b_2^2} \dots\dots (4, 5),$$

$$a_1^2 - a_2^2 = b_1^2 - b_2^2 \dots\dots\dots (6);$$

from which it may be inferred without difficulty that

$$\frac{l_1 l_2}{a_1^2 a_2^2} + \frac{m_1 m_2}{b_1^2 b_2^2} = 0 \dots\dots\dots (7),$$

which, showing that the four tangents at the four points of contact of S with S_1 and S_2 form a rectangular parallelogram, and consequently that the director circle of S coincides with that of the confocal pair S_1 and S_2 , therefore &c.

2. The equations of the three fixed quadrics S_1, S_2, S_3 being

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} = 1, \quad \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} + \frac{z^2}{c_2^2} = 1, \quad \frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} + \frac{z^2}{c_3^2} = 1 \dots\dots (8),$$

that of the variable quadric S , which is evidently concentric with them, is consequently

$$\left[\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} + \frac{z^2}{c_1^2} - 1 \right] - f_1^2 \left[\frac{y}{m_1} - \frac{z}{n_1} \right]^2 - g_1^2 \left[\frac{z}{n_1} - \frac{x}{l_1} \right]^2 - h_1^2 \left[\frac{x}{l_1} - \frac{y}{m_1} \right]^2 = 0 \quad \dots\dots\dots (9),$$

where $x : y : z = l_1 : m_1 : n_1$ is its diameter of double contact with S_1 ; or

$$\left[\frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} + \frac{z^2}{c_2^2} - 1 \right] - f_2^2 \left[\frac{y}{m_2} - \frac{z}{n_2} \right]^2 - g_2^2 \left[\frac{z}{n_2} - \frac{x}{l_2} \right]^2 - h_2^2 \left[\frac{x}{l_2} - \frac{y}{m_2} \right]^2 = 0 \quad \dots\dots\dots (10),$$

where $x : y : z = l_2 : m_2 : n_2$ is its diameter of double contact with S_2 , or

$$\left[\frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} + \frac{z^2}{c_3^2} - 1 \right] - f_3^2 \left[\frac{y}{m_3} - \frac{z}{n_3} \right]^2 - g_3^2 \left[\frac{z}{n_3} - \frac{x}{l_3} \right]^2 - h_3^2 \left[\frac{x}{l_3} - \frac{y}{m_3} \right]^2 = 0 \quad \dots\dots\dots (11),$$

where $x : y : z = l_3 : m_3 : n_3$ is its diameter of double contact with S_3 . Hence, identifying coefficients of x^2 , &c. in equations (9), (10), (11), and remembering that S_1, S_2, S_3 are confocal, we get at once the three groups of equations

$$\left. \begin{aligned} \frac{m_1 n_1}{f_1^2} &= \frac{m_2 n_2}{f_2^2} = \frac{m_3 n_3}{f_3^2} \\ \frac{n_1 l_1}{g_1^2} &= \frac{n_2 l_2}{g_2^2} = \frac{n_3 l_3}{g_3^2} \\ \frac{l_1 m_1}{h_1^2} &= \frac{l_2 m_2}{h_2^2} = \frac{l_3 m_3}{h_3^2} \end{aligned} \right\} \dots\dots\dots (12),$$

$$\left. \begin{aligned} \frac{g_1^2 + h_1^2}{l_1^2} - \frac{1}{a_1^2} &= \frac{g_2^2 + h_2^2}{l_2^2} - \frac{1}{a_2^2} = \frac{g_3^2 + h_3^2}{l_3^2} - \frac{1}{a_3^2} \\ \frac{h_1^2 + f_1^2}{m_1^2} - \frac{1}{b_1^2} &= \frac{h_2^2 + f_2^2}{m_2^2} - \frac{1}{b_2^2} = \frac{h_3^2 + f_3^2}{m_3^2} - \frac{1}{b_3^2} \\ \frac{f_1^2 + g_1^2}{n_1^2} - \frac{1}{c_1^2} &= \frac{f_2^2 + g_2^2}{n_2^2} - \frac{1}{c_2^2} = \frac{f_3^2 + g_3^2}{n_3^2} - \frac{1}{c_3^2} \end{aligned} \right\} \dots\dots\dots (13),$$

$$\left. \begin{aligned} a_2^2 - a_3^2 &= b_2^2 - b_3^2 = c_2^2 - c_3^2 \\ a_3^2 - a_1^2 &= b_3^2 - b_1^2 = c_3^2 - c_1^2 \\ a_1^2 - a_2^2 &= b_1^2 - b_2^2 = c_1^2 - c_2^2 \end{aligned} \right\} \dots\dots\dots (14);$$

from which, as above, it may be inferred without difficulty that

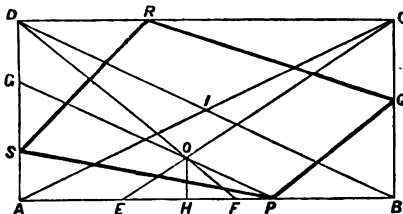
$$\left. \begin{aligned} \frac{l_2 l_3}{a_2^2 a_3^2} + \frac{m_2 m_3}{b_2^2 b_3^2} + \frac{n_2 n_3}{c_2^2 c_3^2} &= 0 \\ \frac{l_3 l_1}{a_3^2 a_1^2} + \frac{m_3 m_1}{b_3^2 b_1^2} + \frac{n_3 n_1}{c_3^2 c_1^2} &= 0 \\ \frac{l_1 l_2}{a_1^2 a_2^2} + \frac{m_1 m_2}{b_1^2 b_2^2} + \frac{n_1 n_2}{c_1^2 c_2^2} &= 0 \end{aligned} \right\} \dots\dots\dots (15),$$

which, showing that the six tangent planes at the six points of contact of S with S_1, S_2, S_3 form a rectangular parallelepiped, and consequently that the director sphere of S coincides with that of the confocal triad S_1, S_2, S_3 therefore, &c.

3521. (Proposed by S. WATSON.)—Within a given square a point is taken at random, and also upon the sides four points are taken at random, one on each. Find the chance that the first point will lie within the quadrilateral formed by joining the other four.

Solution by the PROPOSER.

Instead of a square, take the rectangle ABCD, whose diagonals intersect in I. Let O be the first point; P, Q, R, S the other four. Through O draw CE, DF, PG, meeting BA and AD in E, F, and G; also draw OH perpendicular to AB. Put $AB = a$, $BC = b$, $AH = x$, $HO = y$, $AP = z$.



Then $AE = \frac{bx-ay}{b-y}$, $AF = \frac{bx}{b-y}$, and $AG = \frac{yz}{z-x}$.

Now when P lies on BF, the number of positions of P, Q, R, S, such that O may lie within the quadrilateral PQRS, is

$$ab \int_{AP}^a AG \cdot dz = aby \left\{ a - \frac{bx}{b-y} + x \log \frac{(a-x)(b-y)}{xy} \right\} \dots\dots\dots(1).$$

Similarly, when P lies on AE, the number of positions is

$$aby \left\{ \frac{bx-ay}{b-y} - (a-x) \log \frac{y(a-x)}{x(b-y)} \right\} \dots\dots\dots(2);$$

and when P lies on EF, the number of positions is $ab^2 \cdot EF = \frac{a^2b^2y}{b-y} \dots\dots(3).$

In order that O may lie everywhere in the triangle AIB, the limits are x from $\frac{ay}{b}$ to $\frac{a(b-y)}{b}$, y from 0 to $\frac{1}{2}b$. Hence

$$\int_0^{\frac{1}{2}b} dy \int_{\frac{ay}{b}}^{\frac{a(b-y)}{b}} dx \{ (1) + (2) + (3) \} = \frac{1}{4}a^3b^3;$$

and since the result will be the same when O lies in each of the triangles BIC, CID, DIA, and the total number of the positions of the points taken at random is a^3b^3 , therefore the required chance is $\frac{1}{4}$.

Otherwise: The chance that the point O will lie within the quadrilateral PQRS is (quad. PQRS) $\div ab$. But when P, Q, R, S take all positions on the sides, the average area of the quad. PQRS = $\frac{1}{4}ab$; therefore the required chance is $\frac{1}{4}ab \div ab = \frac{1}{4}$.

3725. (Proposed by C. TAYLOR, M.A.)—The bisectors of the vertical angle A of a triangle meet the base in P, Q; trace the variations of magnitude in $AO : OP$ and $AO : OQ$ as O moves along the base, and apply the result to prove that a conic is concave to its axis.

Now in the hyperbola, $pp = r^2 - a^2 + b^2$, and $\sin \psi = \frac{ab}{rb'} = \frac{ab}{r(r^2 - a^2 + b^2)^{\frac{1}{2}}}$;
but $\frac{ds}{dr} = \frac{1}{\cos \psi}$, and we know that T, the attraction of the rod,

$$= \frac{\mu b}{yb'} = \frac{\mu b}{y(r^2 - a^2 + b^2)^{\frac{1}{2}}};$$

therefore the above equation becomes, after these substitutions,

$$\frac{dF}{dr} - \frac{F}{r} + 2\mu c \frac{r^2}{(r^2 - a^2)(r^2 - a^2 + b^2)(r^2 + b^2)^{\frac{1}{2}}} = 0,$$

the integral of which is $F = r \left\{ c + c' \int \frac{r dr}{(r^2 - a^2)(r^2 - a^2 + b^2)(r^2 + b^2)^{\frac{1}{2}}} \right\}$.

This is easily completed by putting $(r^2 + b^2) = z$; therefore

$$F = r \left\{ c + c' \int \frac{dz}{(z^2 - a^2)(z^2 - c^2)} \right\} = r \left\{ c + c' \log \left(\frac{z-a}{z+a} \right)^{\frac{1}{2}} \cdot \left(\frac{z+c}{z-c} \right)^{\frac{1}{2}} \right\}.$$

If we substitute $(r^2 + b^2)^{\frac{1}{2}}$ for z , we have the expression for the required central repulsive force.

3716. (Proposed by S. TERAY, B.A.)—A small marble is thrown at random on a square table, having an elevated rim. If it be struck at random in any direction, determine the probability that it impinges (1) on two opposite sides; (2) on two adjacent sides, and one opposite; (3) on three consecutive sides; (4) on the four sides in succession.

Solution by the PROPOSER.

Let ABCD be the square, M the marble, Mabcd its path, MN perpendicular to AB. Bisect AB, CD in E, F; and join BF, BD, BM, and EF, cutting BD in O.

Let BA = x , MN = y , AB = 1, Dd = s , $\angle MaN = \theta$. Then we have

$$Na = y \cot \theta,$$

$$Ba = x - y \cot \theta,$$

$$Bb = x \tan \theta - y,$$

$$Cb = y + 1 - x \tan \theta,$$

$$Cc = (y + 1) \cot \theta - x, \quad Dc = x + 1 - (y + 1) \cot \theta,$$

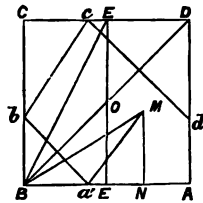
$$(x + 1) \tan \theta - (y + 1) = s;$$

therefore

$$\tan \theta = \frac{y + 1 + s}{x + 1}.$$

In determining the limits of integration, s may vary from $s = 0$ to $s = 1$. The least value of θ is MBN; and in this case the marble will return in the direction BM. When the marble strikes two opposite sides, the least value of θ is $\tan^{-1} \frac{y+1}{x}$; therefore

$$P_1 = \frac{4}{\pi} \int_0^1 \int_0^1 \left\{ \frac{\pi}{2} - \tan^{-1} \frac{y+1}{x} \right\} dx dy.$$



For two adjacent sides and one opposite, M must not be in CBD. The least value of θ is $\tan^{-1} \frac{y}{x}$, and the greatest is obtained by making $s = 0$;

$$\text{therefore } P_2 = \frac{4}{\pi} \int_0^1 \int_0^x \left\{ \tan^{-1} \frac{y+1}{x+1} - \tan^{-1} \frac{y}{x} \right\} dx dy.$$

For three consecutive sides, the summation resolves into four parts, according as M is situated in ABD, CBF, FBO, FOD. For these four spaces we have

$$\begin{aligned} P_3 = & \frac{4}{\pi} \int_0^1 \int_0^x \left\{ \tan^{-1} \frac{y+1}{x} - \tan^{-1} \frac{y+2}{x+1} \right\} dx dy \\ & + \frac{4}{\pi} \int_0^1 \int_{2x}^1 \left\{ \tan^{-1} \frac{y+1}{x} - \tan^{-1} \frac{y}{x} \right\} dx dy \\ & + \frac{4}{\pi} \int_0^1 \int_x^{2x} \left\{ \tan^{-1} \frac{y+1}{x} - \tan^{-1} \frac{y+2}{x+1} \right\} dx dy \\ & + \frac{4}{\pi} \int_0^1 \int_x^1 \left\{ \tan^{-1} \frac{y+1}{x} - \tan^{-1} \frac{y+2}{x+1} \right\} dx dy. \end{aligned}$$

For four successive sides, it will be found that M cannot be situated in CBF. We shall find, in the same way,

$$\begin{aligned} P = & \frac{4}{\pi} \int_0^1 \int_0^x \left\{ \tan^{-1} \frac{y+2}{x+1} - \tan^{-1} \frac{y+1}{x+1} \right\} dx dy \\ & + \frac{4}{\pi} \int_0^1 \int_x^{2x} \left\{ \tan^{-1} \frac{y+2}{x+1} - \tan^{-1} \frac{y}{x} \right\} dx dy \\ & + \frac{4}{\pi} \int_0^1 \int_{1-x}^1 \left\{ \tan^{-1} \frac{y+2}{x+1} - \tan^{-1} \frac{y}{x} \right\} dx dy. \end{aligned}$$

The expressions for P_3 and P_4 may be put in the forms

$$\begin{aligned} \frac{1}{4} \pi P_3 = & \int_0^1 \int_0^1 \tan^{-1} \frac{y+1}{x} dx dy - \int_0^1 \int_{2x}^1 \tan^{-1} \frac{y}{x} dx dy \\ & + \left\{ \int_0^1 \int_0^{2x} + \int_{\frac{1}{4}}^1 \int_0^1 \right\} \tan^{-1} \frac{y+2}{x+1} dx dy, \\ \frac{1}{4} \pi P_4 = & \left\{ \int_0^1 \int_0^{2x} + \int_0^1 \int_{\frac{1}{4}}^1 \right\} \tan^{-1} \frac{y+2}{x+1} dx dy - \int_0^1 \int_0^x \tan^{-1} \frac{y+1}{x+1} dx dy \\ & - \left\{ \int_0^1 \int_0^{2x} - \int_0^1 \int_0^x + \int_0^1 \int_{\frac{1}{4}}^1 \right\} \tan^{-1} \frac{y}{x} dx dy; \end{aligned}$$

$$\begin{aligned} \text{therefore } P_1 + P_2 + P_3 + P_4 = & \frac{4}{\pi} \left\{ \frac{\pi}{2} - \int_0^1 \int_0^1 \tan^{-1} \frac{y}{x} dx dy \right\} \\ = & \frac{4}{\pi} \left\{ \frac{\pi}{2} - \frac{\pi}{4} \right\} = 1. \end{aligned}$$

This tests the accuracy of the above limits. We have thus only to calculate the values of P_1 , P_2 , l_3 .

Now $\int \tan^{-1} \frac{y}{x} dx dy = y \tan^{-1} \frac{y}{x} - \frac{1}{2} x \log(x^2 + y^2)$.

By the aid of this formula we find

$$P_1 = \frac{4}{\pi} \left\{ \frac{3}{2}\pi - 2 \tan^{-1} 2 + 2 \log 2 - \frac{3}{2} \log 5 \right\},$$

$$P_2 = \frac{4}{\pi} \left\{ 2 \tan^{-1} 2 - \frac{1}{2}\pi - \frac{3}{2} \log 2 + \frac{3}{2} \log 5 \right\},$$

$$P_3 = \frac{4}{\pi} \left\{ 3 \tan^{-1} 2 - \frac{3}{2}\pi + 6 \tan^{-1} \frac{4}{3} - 6 \log 2 + \frac{9}{2} \log 3 + \frac{1}{2} \log 5 - \frac{3}{2} \log 13 \right\};$$

or $P_1 = .4407, P_2 = .1498, P_3 = .2242, P_4 = .1853$.

3751. (Proposed by the EDITOR.)—Two points taken at random on the surface of a sphere are joined to each other and to a third point taken at random either within or on the surface; or again, the centre of a sphere is joined to two points taken at random, either one or both within or on the surface. Find, in each case, the probability that the triangle thus formed is acute; also find the analogous probabilities in a circle.

Solution by STEPHEN WATSON.

1. Let O (Fig. 1) be the centre of the sphere; P and Q two points on its surface; C the centre of the sphere on PQ as diameter; PA a diameter; and PD, QA tangent planes to the sphere (C) at P and Q.

The several probabilities will be independent of OP, therefore put OP=1, $\angle POC = \theta$. The triangle PQR will be acute when R lies between the planes PD, QA, but without the sphere (C); the number of its positions is therefore

$$\frac{1}{2}\pi(6 \sin \theta - 4 \sin^3 \theta + 3 \cos \theta - \cos^3 \theta - 2) \dots (1).$$

It is plain that P may be taken as fixed. An element of the surface at Q is $2\pi \cdot PQ \cos \theta \cdot d(2\theta) = 8\pi \sin \theta \cos \theta d\theta$, and the whole number of positions of Q and R is $= \frac{1}{2}\pi \cdot 4\pi = 2\pi^2$; hence the required probability

$$\text{in this case is } \frac{3}{2\pi} \int_0^{\frac{1}{2}\pi} (1) \sin \theta \cos \theta d\theta = \frac{1}{2}.$$

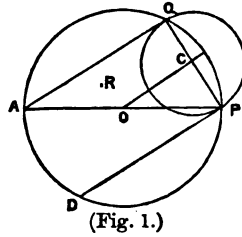
The corresponding probability in the circle is

$$\frac{2}{\pi^2} \int_0^{\frac{1}{2}\pi} (3 \sin \theta \cos \theta + \theta - \frac{1}{2}\pi \sin^2 \theta) d\theta = \frac{3}{\pi^2}.$$

2. When R lies on the surface, (1) becomes

$$2\pi(2 \sin \theta + \cos \theta - 1) \dots \dots \dots (2),$$

and the number of positions of Q and R $= (4\pi)^2$; hence the probability in



this case is
$$\frac{1}{2\pi} \int_0^{2\pi} (2) \sin \theta \cos \theta \, d\theta = \frac{1}{2}.$$

In the circle the corresponding probability is

$$\frac{2}{\pi} \int_0^{\frac{1}{2}\pi} \theta \, d\theta = \frac{1}{4}.$$

3. When one point P is at the centre, and Q, R within the sphere. Let (O) be the sphere (Fig. 2) on PQ as diameter; and CD, EF tangent planes at P and R. Then R must lie between those planes, but without the sphere (O), and the number of its positions is, putting $\angle EPQ = \theta$,

$$\frac{1}{2}\pi (2 - \cos^2 \theta) \cos \theta \dots\dots\dots (3);$$

hence the probability required in this case is

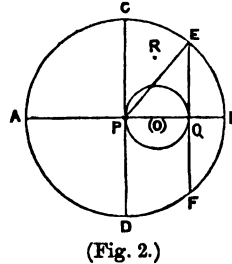
$$\frac{9}{16\pi^2} \int_{\frac{1}{2}\pi}^0 (3) 4\pi \cos^2 \theta \, d\theta = \frac{3}{8}.$$

The corresponding probability in the circle is

$$\frac{1}{\pi} \int_{\frac{1}{2}\pi}^0 (\pi - 2\theta + 2 \sin \theta \cos \theta - \frac{1}{2}\pi \cos^2 \theta) \cos \theta \, d(\cos \theta) = \frac{1}{4}.$$

4. When Q is on the surface, it may be taken as fixed. Then putting $\theta = 0$ in (3), and dividing the result by $\frac{1}{2}\pi$, we have the probability in this case = $\frac{3}{8}$. In the circle = $\frac{1}{4}$.

5. When R is also on the surface. For an acute triangle, R may lie on the semi-surface, and the chance is $\frac{1}{4}$. In the circle, the same.



(Fig. 2.)

DESCRIPTION OF AN INSTRUMENT FOR DRAWING QUARTIC BINODAL CURVES. By S. ROBERTS, M.A.

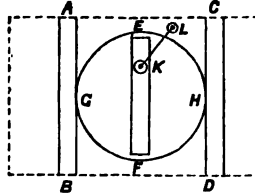
A point rigidly connected with a link of given length, the extremities of which move respectively on a right line and a conic, describes a binodal quartic curve, which is circular when the conic is a circle; and I have elsewhere mentioned certain obvious and simple methods of tracing such curves with more or less accuracy.

It is, however, mechanically advantageous to fix the pencil and make the drawing-board moveable. The construction is then equivalent to moving a given angle with one of its arms through a fixed point, while the extremity of the other arm, which is of given length, moves on a fixed circle.

A pencil rigidly connected with the moving angle would describe a curve of the 6th degree, which breaks up into a Limaçon and a circle if the fixed point is on the circle. Conversely, the pencil being fixed, the locus described on the moving plane will be a quartic binodal curve, becoming an ellipse when the circle passes through the fixed point.

We have then the following construction :—

Two parallel slots AB, CD are fixed to the under side of the drawing-board ABCD, and slip about the fixed circular board EFGH. This board has an aperture EF, in which the pivot K may be clamped. The link KL moves about K and about L, a point fixed to the drawing-board. Within limits, we may vary the length KL and the positions of K and L. By this means we can trace, with a fixed pencil, quartic curves passing into a great variety of instructive forms, and the instrument is of simple construction. It will be observed that there is a point-to-point correspondence between the curves described by the fixed and moving pencils.



3749. (Proposed by Professor CROFTON, F.R.S.)—An inflexible horizontal bar is suspended by three equal and similar vertical rods, attached at its middle and ends; a given weight being attached by a hook to any given point of the lower surface of the bar, determine the tensions of the three rods; show that, for certain positions of the weight, one of the rods will be compressed (as may be verified by experiment).

Solution by Professor TOWNSEND, F.R.S.

Denoting by A and B the two extremities, by C the middle point, by D the centre of gravity, and by W the weight of the bar; by X the point of suspension of the appended weight W', supposed to be at the same side of C with A and D; by P, Q, and R the tensions of the three supporting rods at A, B, and C respectively; and by a , d , and x the three intervals CA, CD, and CX respectively. Then, on elementary principles of statics, for the equilibrium of the system,

$$P + Q + R = W + W', \quad (P - Q)a = Wd + W'x \dots\dots\dots (1, 2),$$

which, if the supporting rods were perfectly rigid, would be the only equations connecting the three unknown tensions P, Q, R, which, as is well known, would in that case be consequently indeterminate.

The elasticity of the rods, combined with the inflexibility of the bar, furnishes at once the third equation requisite for their determination; for, while it follows from the former that the tension of each rod is proportional to its extension, it follows also from the latter that the sum of the extensions of the extreme rods is equal to twice the extension of the mean rod; hence

$$P + Q = 2R \dots\dots\dots (3),$$

which accordingly, in the present case, is the third equation required.

Solving from (1), (2), and (3), for P, Q, and R, we get at once

$$P = \frac{W(2a + 3d) + W'(2a + 3x)}{6a}, \quad Q = \frac{W(2a - 3d) + W'(2a - 3x)}{6a},$$

$$R = P + Q = \frac{1}{3}(W + W') \dots\dots\dots (4).$$

from which it appears that, while P and R are always necessarily positive, Q on the contrary may be, and sometimes is, negative.

If W may be neglected in comparison with W' , then is $x = \frac{2}{3}a$ the limit within which Q is positive, at which it vanishes, and beyond which it is negative.

If, in the proposed question, the two intervals CA and CB had been given unequal, then, denoting them by a and b respectively, the above equations (2) and (3) would have been respectively

$$Pa - Qb = Wd + W'x, \quad Pb + Qa = R(b + a) \dots\dots (2', 3'),$$

which, with (1), would have been the equations for the determination of the three tensions.

3758. (Proposed by J. J. WALKER, M.A.)—Determine y as a function of x from

$$3 \left\{ \frac{d \frac{y}{v}}{dx} \right\}^2 + \frac{d \left(\frac{u}{v} \right)^3}{dx} \cdot \frac{d \frac{y}{u}}{dx} = 0,$$

u and v being given functions of x only.

Solution by the PROPOSER.

If y', v', u' stand for $\frac{dy}{dx}, \frac{dv}{dx}, \frac{du}{dx}$ respectively, the equation above gives

$$(vy' - v'y)^2 + (vu' - v'u)(uy' - u'y) = 0,$$

which is the result of the elimination of a between

$$va^2 + ua - y = 0 \quad \text{and} \quad v'a^2 + u'a - y' = 0.$$

The latter of these two being the derivative of the former with respect to x , a being considered as constant, $y = au + a^2v$ is the complete integral of the given equation.

3436. (Proposed by J. J. WALKER, M.A.)—Tangents are drawn from a point (x', y') to the semi-cubic parabola $x^3 - 3ay = 0$: show that the circle passing through the points of contact has for its equation

$$ax'(x^2 + y^2) - 3x'^2(x + a)x + 2ay'(2x' + a)y + 4a(x' + a)y'^2 = 0;$$

also determine and trace the locus of (x', y') when the circle touches the parabola at a point which is *not* one of the points of contact of the tangents from (x', y') .

Solution by R. TUCKER, M.A.

Adopting the mode of solution indicated in Mr. Walker's paper "On Tangents to a Cissoid," read before the London Mathematical Society (see also Quest. 2918, *Reprint*, Vol. XIII., p. 87), we have for determining

the circle in question, the equations

$$x^2 - 3ay^2 = 0, \quad x'^2 - 2ayy' - ay^2 = 0 \dots\dots\dots (1, 2),$$

which are equivalent to (2), and $3x'y - 2xy' - xy = 0 \dots\dots\dots (3)$;

that is, to $x'^2 = ay(2y' + y)$, $3x'y = x(2y' + y)$.

Multiplying these together, we get $3x'^2x = a(4y'^2 + 4y'y + y^2)$;

that is, $ay^2 + 4ayy' + 4ay'^2 - 3x'^2x = 0 \dots\dots\dots (4)$.

Substituting for ay^2 in (2), we have

$$x'^2 + 2ayy' + 4ay'^2 - 3x'^2x = 0 \dots\dots\dots (5).$$

Multiply (4) by x' and (5) by a , and add, and we obtain for the equation to the circle $ax'(x^2 + y^2) + 2a(2x' + a)y'y - 3x'^2(x' + a)x + 4a(x' + a)y^2 = 0$.

I get for the locus (by means of the discriminant) of (x', y') in the case supposed $9x'^5(x' + a)^3 = 8a^4y'^2(2x' + a)^2$

of rather a high degree.

Since $3x'^2 - x^2 = 6ayy'$, squaring and substituting, we get

$$x^3 - 6x'^2x^2 + 9x'^2x - 12ay'^2 = 0 \dots\dots\dots (\alpha);$$

and from $x'^2 = ay(2y' + y)$, by cubing and substituting, we get

$$ay^3 + 6ay'y^2 + (12ay'^2 - 9x'^2)y + 8ay'^3 = 0 \dots\dots\dots (\beta).$$

The equations (α) and (β) give the abscissæ and ordinates of the points of contact, and we find $x_1 + x_2 + x_3 = 6x'$, $y_1 + y_2 + y_3 = -6y'$; hence the locus of the centroid of the triangle formed by joining the points is a curve similar to that of (x', y') .

3766. (Proposed by R. TUCKER, M.A.)—If ρ_1, ρ_2, ρ_3 be the inscribed radii of the triangles $D_1E_1F_1, D_2E_2F_2, D_3E_3F_3$, formed by joining the points of contact of each of the escribed circles with the three sides BC, CA, AB of a triangle ABC; prove that

$$\frac{1}{\rho_1} : \frac{1}{\rho_2} : \frac{1}{\rho_3} = 1 - \tan \frac{1}{2}A : 1 - \tan \frac{1}{2}B : 1 - \tan \frac{1}{2}C.$$

I. Solution by A. ESCOTT, F.R.A.S.; J. H. SLADE; Q. W. KING; and others.

Draw the lines as in the figure.

$$\angle D_1F_1E_1 = \frac{1}{2}D_1C_1E_1 = \frac{1}{2}C, \quad \angle D_1E_1F_1 = \frac{1}{2}B,$$

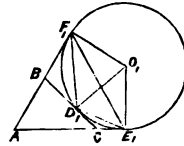
$$\angle F_1D_1E_1 = 90^\circ + \frac{1}{2}A,$$

$$D_1E_1 = 2D_1O_1 \cdot \sin \frac{1}{2}D_1C_1E_1 = 2r_1 \cdot \sin \frac{1}{2}C,$$

$$D_1F_1 = 2r_1 \cdot \sin \frac{1}{2}B,$$

$$F_1E_1 = 2r_1 \cdot \sin \frac{1}{2}F_1O_1E_1 = 2r_1 \sin \frac{1}{2}(180^\circ - A) \\ = 2r_1 \cdot \cos \frac{1}{2}A.$$

$$\text{Now} \quad \rho_1 = \frac{1}{2}(D_1E_1 + D_1F_1 - F_1E_1) \cdot \tan \frac{1}{2}F_1D_1E_1 \\ = r_1(\sin \frac{1}{2}B + \sin \frac{1}{2}C - \cos \frac{1}{2}A) \cdot \tan (45^\circ + \frac{1}{2}A);$$



$$\begin{aligned}
 \therefore \rho_1(1 - \tan \tfrac{1}{2}A) &= s \tan \tfrac{1}{2}A (\sin \tfrac{1}{2}B + \sin \tfrac{1}{2}C - \cos \tfrac{1}{2}A)(1 + \tan \tfrac{1}{2}A) \\
 &= 2 \sqrt{2} \cdot s \sin \tfrac{1}{2}A \cdot \sin \tfrac{1}{2}B \cdot \sin \tfrac{1}{2}C \\
 &= \rho_2(1 - \tan \tfrac{1}{2}B) = \rho_3(1 - \tan \tfrac{1}{2}C);
 \end{aligned}$$

$$\text{therefore } \frac{1}{\rho_1} : \frac{1}{\rho_2} : \frac{1}{\rho_3} = 1 - \tan \tfrac{1}{2}A : 1 - \tan \tfrac{1}{2}B : 1 - \tan \tfrac{1}{2}C.$$

II. *Solution by Q. W. KING; J. H. SLADE; S. WATSON;
T. MITCHESON, B.A.; and others.*

It is easily proved that $\angle D_1E_1F_1 = \tfrac{1}{2}B$ and
 $\angle D_1F_1E_1 = \tfrac{1}{2}C$; hence

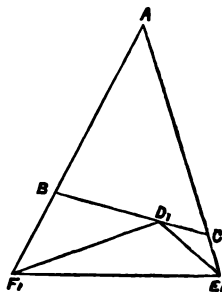
$$\rho_1(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C) = E_1F_1 = 2s \sin \tfrac{1}{2}A.$$

$$\text{Similarly } \rho_2(\cot \tfrac{1}{2}C + \cot \tfrac{1}{2}A) = 2s \sin \tfrac{1}{2}B;$$

$$\text{therefore } \frac{1}{\rho_1} : \frac{1}{\rho_2} =$$

$$\begin{aligned}
 (\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C) \sin \tfrac{1}{2}B &: (\cot \tfrac{1}{2}C + \cot \tfrac{1}{2}A) \sin \tfrac{1}{2}A \\
 &= \sin \tfrac{1}{2}(\pi - A) \cos \tfrac{1}{2}B : \sin \tfrac{1}{2}(\pi - B) \cos \tfrac{1}{2}A \\
 &= 1 - \tan \tfrac{1}{2}A : 1 - \tan \tfrac{1}{2}B.
 \end{aligned}$$

Hence, by symmetry, the whole result required is obtained.

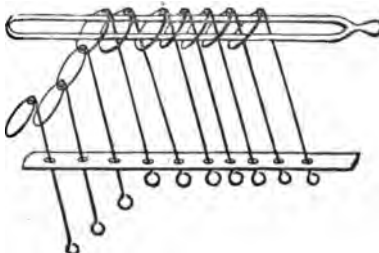


3575. (Proposed by G. S. CARR.)—In a toy called Chinese Rings, a number of rings are hung upon a bar in such a manner that the ring at one end (A) can be put on or off the bar at pleasure; but any other ring can only be put on or off when the next one to it towards (A) is on, and all the rest towards (A) off the bar. The order of the rings cannot be changed. Show that if there are sixty rings, it will be necessary, in order to disconnect them from the bar, to put a ring either on or off 768614336404564650 times.

Solution by the PROPOSER.

Let the putting a ring either on or off the bar be termed a movement.

When $n-1$ rings are disconnected, let M be the number of movements requisite for taking off the n th and $(n+1)$ th rings. We will show that $4M$ movements will afterwards take off the $(n+2)$ th and $(n+3)$ th rings.



The $(n+3)$ th is taken off in 1 movement.
 The n th is put on, the $(n+1)$ th is put on, and the } in $2M-1$ movements.
 n th is taken off }
 The $(n+2)$ th is taken off in 1 movement.
 The n th is put on, the $(n+1)$ th is taken off, and } in $2M-1$ movements.
 the n th is taken off }
 Total number of movements in disconnecting the } = $4M$.
 $(n+2)$ th and $(n+3)$ th rings }

Now let $n=1$, $M=2$;

therefore 1st and 2nd rings are taken off in 2 movements,
 then the 3rd and 4th " " 2^2 "
 then the 5th and 6th " " 2^3 "
 2^{n-1} "
 lastly, the $(n-1)$ th and n th " " 2^{n-1} "
 Total number of movements } = $2 + 2^2 + 2^3 + \dots + 2^{n-1} = \frac{2^{n+1} - 2}{3}$.
 in taking off n rings }

Similarly, if n be odd, by taking off the 1st ring, then the 3rd, and so on,
 the number of movements = $\frac{1}{3}(2^{n+1} - 1)$.

If $n=60$, we have $\frac{1}{3}(2^{61} - 2) = 768614336404564650$.

3764. (Proposed by B. WILLIAMSON, M.A.)—Prove that the ordinary solution for the position when a planet appears brightest gives no real position if the planet be nearer to the sun than one-fourth of the earth's distance.

Solution by the PROPOSER.

Let a and r denote respectively the distances of the earth and planet from the sun, x the distance at which the planet appears brightest; then, by the ordinary method, we get $x = (3a^2 + r^2)^{\frac{1}{2}} - 2r$ (Godfray's *Astronomy*, p. 276). Now, since x must be less than $a+r$, we must have $3r+a > (3a^2 + r^2)^{\frac{1}{2}}$; hence $4r^2 + 3ar > a^2$, or $2r + \frac{3}{4}a > \frac{1}{2}a$, therefore $r > \frac{1}{4}a$. It is easily seen that the real maximum takes place when the planet is behind the sun.

3801. (Proposed by R. W. GENESE, B.A.)—ABCD is a parallelogram inscribed in a rectangular hyperbola. Prove (1) that AB, CD opposite sides subtend either equal or supplementary angles at any point P of the curve; (2) that the circles circumscribing PAB, PBC, PCD, PDA, are equal; and (3) that the product of the perpendiculars from P on each pair of opposite sides is the same.

I. Solution by C. TAYLOR, M.A.

The first part of this question is a particular case of the theorem that "The angle between any two chords is equal to that between their supplements" (*Messenger of Mathematics*, Vol. V., p. 170). The following form of statement may be suggested:—

Any two arcs diametrically opposite subtend equal angles at the curve.

The following are statements of known theorems:—

(i.) Any arc subtends equal angles at points on the curve diametrically opposite.

(ii.) Any chord subtends equal or supplementary angles at the ends of a perpendicular chord.

II. Solution by JAMES DALE.

Let ABCD be the parallelogram, and P, Q any other two diametrically opposite points on the hyperbola; complete the parallelograms APCQ, BPDQ.

BA, DA being supplemental chords are parallel to conjugate diameters, and hence are equally inclined to an asymptote; so also are QA, PA; and therefore $\angle BAP = \angle QAD = \angle QCD$, and $\angle PAD = \angle BAQ = \angle BCQ$. For the same reason $\angle ABP = \angle CBQ = \angle CDQ$, and $\angle ABQ = \angle CBP = \angle CDP$. When two points such as P, Q lie within the quadrilateral formed by the other four, then $\angle APB + \angle CPD = 180^\circ$, and therefore $\angle APD + \angle CPB = 180^\circ$.

From the equalities of the angles, it readily follows that the circles PAB, PBC, PCD, PDA are equal.

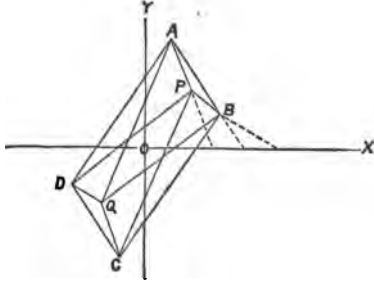
The product of the perpendiculars from P on the pair AB, CD

$$= PA \cdot PC \cdot \sin \angle PAB \sin \angle PCD,$$

so also the product on the pair AD, BC

$$= PA \cdot PC \cdot \sin \angle PAD \sin \angle PCB = PA \cdot PC \cdot \sin \angle PCD \sin \angle PAB.$$

The same property holds good with respect to A and the parallelogram BPDQ, also with B and the parallelogram APCQ.



3744. (Proposed by the Rev. H. T. SHARPE, M.A.)—To a smooth horizontal plane is fastened a hoop of radius c , which is rough inside, μ being the coefficient of friction. In contact with this, a disc of radius a is spun with initial angular velocity n , and its centre is projected with velocity v in such a direction as to be most retarded by friction. Show that after a time $\frac{c-a}{\mu} \frac{k^2}{v} \frac{an+v}{a^2v - ank^2}$, the disc will roll inside the hoop.

Solution by X. Y. Z.

Let u be the velocity of the centre of gravity, and ω the angular velocity. Resolve parallel to the tangent at any moment, and take moments about a fixed point coincident with the point of contact at that moment.

$$\text{Since friction} = \frac{\mu u^2}{c-a}, \quad \frac{du}{dt} = -\frac{\mu u^2}{c-a}, \quad \frac{1}{u} - \frac{1}{v} = \frac{\mu t}{c-a};$$

also

$$au - k^2\omega = av - k^2n;$$

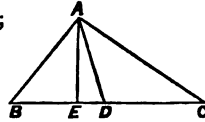
and we shall have $av + u = 0$ when

$$u = \frac{a}{a^2 + k^2} (av - k^2n), \quad \text{that is, when} \quad t = \frac{c-a}{\mu} \frac{k^2}{v} \frac{an+v}{a^2v - ank^2}.$$

3791. (Proposed by T. MITCHELSON, B.A.)—If D be the middle point of the base BC of a triangle ABC, and E the foot of the perpendicular from the vertical angle; prove that $2BC \cdot DE = AC^2 - AB^2$.

*Solution by W. PRIESTLEY; J. H. SLADE; Q. W. KING;
W. SIVERLEY; and many others.*

$$\begin{aligned} AC^2 - AB^2 &= CE^2 - BE^2 = BC(CE - BE) \\ &= 2BC \cdot DE. \end{aligned}$$



3724. (Proposed by M. COLLINS, B.A.)—If N be a given number, x^3 the nearest integral cube number $= N + D$; prove that $N^{\frac{1}{3}}$, which is $= x$ nearly, and $= x + d$ exactly,

$$\begin{aligned} &= \frac{x(7N^3 + 105N^2x^3 + 120Nx^6 + 11x^9) + 21d^5x^2(N - x^3) - d^7(10x^3 - N)}{N^3 + 60N^2x^3 + 147Nx^6 + 35x^9} \text{ exactly} \\ &= \frac{(243N^3 + 378N^2D + 163ND^2 + 11D^3)x - 21Dx^2d^5}{243N^3 + 459N^2D + 252ND^2 + 35D^3} \text{ very nearly} \\ &= \frac{3N(9x^3 - 2D)^2 + D^2(6x^3 + 5D)}{3N(18x^3 - D)^2 + 141D^2x^3 - D^3} \times 4x \text{ nearly.} \end{aligned}$$

Solution by N'IMPORTE.

Since $(N^{\frac{1}{3}} - x)^7 = d^7$ and $(N^{\frac{1}{3}} - x)^6 = d^6$, therefore

$$\begin{aligned} 21d^5x^3(N - x^3) - d^7(10x^3 - N) &= 21x^3(N - x^3)(N^{\frac{1}{3}} - x)^6 - (10x^3 - N)(N^{\frac{1}{3}} - x)^7 \\ &= 11x^{10} - 35x^9N^{\frac{1}{3}} + 120x^7N - 147x^6N^{\frac{2}{3}} + 105x^5N^{\frac{4}{3}} - 60x^4N^{\frac{5}{3}} + 7x^3N^{\frac{2}{3}} - N^{\frac{7}{3}}. \end{aligned}$$

Hence, transposing, we obtain for $N^{\frac{1}{3}}$ the exact value given in the question.

Putting $N + D$ for x^3 in this result, and neglecting d^7 ,

we obtain $N^{\frac{1}{2}} = 1\text{st approximation given;}$
and neglecting also $d^{\frac{1}{2}}$, $N^{\frac{1}{2}} = 2\text{nd approximation.}$

[For example, let $N=29$, and take $x=3$, $D = x^2 - N = -2$; then the last expression gives

$$29^{\frac{1}{2}} = \frac{87(247)^2 + 4(152)}{87(488)^2 + 141 \times 4 \times 27 + 8} \times 12 = 3.0723168,$$

which is correct even to its last decimal figure.]

3312. (Proposed by Dr. J. MATTESON.)—It is required to find four square numbers such that (1) the sum of their roots shall be a square, (2) the sum of the first and fourth, (3) the sum of the second and fourth, (4) the sum of the roots of the third and fourth multiplied by the root of the third, or (5) by the root of the sum of the second and fourth, (6) the sum of the first three added to three times the third, and (7) the sum of all four added to three times the third,—shall all be squares; (8) the product of the third and fourth added to the product of the first and second, (9) to the product of the first and third, (10) to the product of the second and third, or (11) to the sum of these three products,—shall produce squares; (12) any one of the numbers added to the sum of the roots of the other three, (13) the square of the sum of any two of their roots added to the sum of the other two roots, or (14) the square of the sum of any three of their roots added to the remaining root,—a square; (15) also, four other square numbers such that either of the first three added to the sum of the roots of the other three shall all be squares, and which also shall fulfil conditions (2, 3, 4, 5, 6, 7, 8, 9, 10, 11).

Solution by the PROPOSER; and R. DAVIS.

Let a^2x^2 , b^2x^2 , c^2x^2 , e^2x^2 denote the first set of four squares; and $a^2t^2x^2$, $b^2t^2x^2$, $c^2t^2x^2$, $e^2t^2x^2$ the four square numbers in (15).

Put $a^2 = 4(pq^2 + q^4)$, $b^2 = 2pq^2 + q^4$, $c^2 = q^4$, $e^2 = p^2$; then we must have

$$4(pq^2 + q^4)x^2 = \square \dots [a], \quad (2pq^2 + q^4)x^2 = \square \dots [b],$$

$$\begin{aligned} [1] \quad & 2x(pq^2 + q^4)^{\frac{1}{2}} + x(2pq^2 + q^4)^{\frac{1}{2}} + q^2x + px = \square, \\ [2] \quad & (p^2 + 4pq^2 + 4q^4)x^2 = \square, \quad [7] \quad (p^2 + 6pq^2 + 9q^4)x^2 = \square, \\ [3] \quad & (p^2 + 2pq^2 + q^4)x^2 = \square, \quad [8] \quad (9p^2q^4 + 12pq^6 + 4q^8)x^4 = \square, \\ [4] \quad & (p + q^2)x \times q^2x = \square, \quad [9] \quad (p^2q^4 + 4pq^6 + 4q^8)x^4 = \square, \\ [5] \quad & (p + q^2)x \times (p + q^2)x = \square, \quad [10] \quad (p^2q^4 + 2pq^6 + q^8)x^4 = \square, \\ [6] \quad & (6pq^2 + 9q^4)x^2 = \square, \quad [11] \quad (9p^2q^4 + 18pq^6 + 9q^8)x^4 = \square. \end{aligned}$$

These are all squares except $[a]$, $[b]$, $[1]$, $[4]$, $[6]$, which reduce to $p + q^2 = \square$, $2p + q^2 = \square$, $[2q(p + q^2)^{\frac{1}{2}} + q(2p + q^2)^{\frac{1}{2}} + q^2 + p]x = \square$, and $6p + 9q^2 = \square$; $[a]$ and $[4]$ being identical when reduced.

Since q^2 , $p + q^2$, and $2p + q^2$ will be three square numbers in arithmetical progression, we may take $q = r^2 - 2rs - s^2$ and $p = 4rs(r^2 - s^2)$; then will

$$\begin{aligned}
a^2x^2 &= 4(r^2 - 2rs - s^2)^2(r^2 + s^2)^2x^2 = 4(pq^2 + q^4)x^2, \\
b^2x^2 &= (r^2 + 2rs - s^2)^2(r^2 - 2rs - s^2)^2x^2 = (2pq^2 + q^4)x^2, \\
c^2x^2 &= (r^2 - 2rs - s^2)^4x^2 = q^4x^2, \quad e^2x^2 = 16r^2s^2(r^2 - s^2)^2x^2 = p^2x^2.
\end{aligned}$$

Substituting the above values of p and q in [6], it becomes
 $9r^4 - 12r^3s + 18r^2s^2 + 12rs^3 + 9s^4 = \square$, say $= (3r^2 - 2rs + 3s^2)^2$; then $r = 6s$.
 Therefore $4(pq^2 + q^4)x^2 = (1702)^2s^2x^2$, $(2pq^2 + q^4)x^2 = (1081)^2s^2x^2$,
 $q^4x^2 = (23)^4s^2x^2$, $p^2x^2 = (840)^2s^2x^2$.

To satisfy the first condition of the problem, assume

$$(a + b + c + e)x = m^2; \text{ then } x = m^2 \div (a + b + c + e).$$

Premising that either the fractional or integral expressions in the first four of the fourteen following lines are the squares in [12]; the next six, those in [13]; the last four, those in [14]; the second expression in each line being produced by multiplying each term in the first expression by $(a + b + c + e)^2 \div m^2$, we have

$$\begin{aligned}
&\frac{a^2m^4}{(a+b+c+e)^2} + \frac{(b+c+e)m^2}{a+b+c+e}, \text{ or } a^2m^2 + a(b+c+e) + (b+c+e)^2, \\
&\frac{b^2m^4}{(a+b+c+e)^2} + \frac{(a+c+e)m^2}{a+b+c+e}, \text{ or } b^2m^2 + b(a+c+e) + (a+c+e)^2, \\
&\frac{c^2m^4}{(a+b+c+e)^2} + \frac{(a+b+e)m^2}{a+b+c+e}, \text{ or } c^2m^2 + c(a+b+e) + (a+b+e)^2, \\
&\frac{e^2m^4}{(a+b+c+e)^2} + \frac{(a+b+c)m^2}{a+b+c+e}, \text{ or } e^2m^2 + e(a+b+c) + (a+b+c)^2, \\
&\frac{(a+b)^2m^4}{(a+b+c+e)^2} + \frac{(c+e)m^2}{a+b+c+e}, \text{ or } (a+b)^2m^2 + (a+b)(c+e) + (c+e)^2, \\
&\frac{(a+c)^2m^4}{(a+b+c+e)^2} + \frac{(b+e)m^2}{a+b+c+e}, \text{ or } (a+c)^2m^2 + (a+c)(b+e) + (b+e)^2, \\
&\frac{(a+e)^2m^4}{(a+b+c+e)^2} + \frac{(b+c)m^2}{a+b+c+e}, \text{ or } (a+e)^2m^2 + (a+e)(b+c) + (b+c)^2, \\
&\frac{(b+c)^2m^4}{(a+b+c+e)^2} + \frac{(a+e)m^2}{a+b+c+e}, \text{ or } (b+c)^2m^2 + (b+c)(a+e) + (a+e)^2, \\
&\frac{(b+e)^2m^4}{(a+b+c+e)^2} + \frac{(a+c)m^2}{a+b+c+e}, \text{ or } (b+e)^2m^2 + (b+e)(a+c) + (a+c)^2, \\
&\frac{(c+e)^2m^4}{(a+b+c+e)^2} + \frac{(a+b)m^2}{a+b+c+e}, \text{ or } (c+e)^2m^2 + (c+e)(a+b) + (a+b)^2, \\
&\frac{(a+b+c)^2m^4}{(a+b+c+e)^2} + \frac{em^2}{a+b+c+e}, \text{ or } (a+b+c)^2m^2 + (a+b+c)e + e^2, \\
&\frac{(a+b+e)^2m^4}{(a+b+c+e)^2} + \frac{cm^2}{a+b+c+e}, \text{ or } (a+b+e)^2m^2 + (a+b+e)c + c^2, \\
&\frac{(a+c+e)^2m^4}{(a+b+c+e)^2} + \frac{bm^2}{a+b+c+e}, \text{ or } (a+c+e)^2m^2 + (a+c+e)b + b^2, \\
&\frac{(b+c+e)^2m^4}{(a+b+c+e)^2} + \frac{am^2}{a+b+c+e}, \text{ or } (b+c+e)^2m^2 + (b+c+e)a + a^2.
\end{aligned}$$

To find m , put $a^2m^2 + (b+c+e)a + (b+c+e)^2 = (am+b+c+e)^2$; then will $m = \frac{1}{4}$, and it is obvious that the same value of m will fulfil all the remaining conditions.

$$\text{Therefore } x = \frac{m^2}{a+b+c+e} = \frac{1}{4(1702+1081+529+840)s^4} = \frac{1}{16608s^4};$$

$$\text{and } ax = \frac{1702s^4}{16608s^4} = \frac{1702}{16608}, \quad bx = \frac{1081}{16608}, \quad cx = \frac{529}{16608}, \quad ex = \frac{840}{16608}.$$

To fulfil the first three conditions of (15), let $e' = 840$; and write 1702 for a , 1081 for b , 529 for c , (1081+529+840) for d , (1702+529+840) for e , (1702+1081+840) for f ; then we must have

$$a^2tx^2 + dtx = \square, \quad b^2tx^2 + etx = \square, \quad \text{and} \quad c^2tx^2 + ftx = \square.$$

Solving these equalities, we obtain

$$tx = \frac{(7254600499424020785286419021924)^2}{100346727875645447201875013690942472664962600265504007844014833390081};$$

and writing A^2 for the square numerator of tx , B for its denominator, and C, D, E for the denominators of atx, btx, etx respectively, we find

$$a^2t^2x^2 = \left(\frac{A^2 - 52629227107936646829056079300133415061195064329236799900792661776}{5895812448621941668735312202768077124850916584341985124795231104} \right)^2,$$

$$b^2t^2x^2 = \left(\frac{A^2}{9282768636128163478434321340512717175297187813645144111379725568} \right)^2,$$

$$c^2t^2x^2 = \left(\frac{A^2}{18969135704261899282017961000178161184302949010492251010210743552} \right)^2,$$

$$e^2t^2x^2 = \left(\frac{A^2}{B} \right)^2 = \left(\frac{4420 \&c.}{B} \right)^2 \\ = \left(\frac{4420855077066663336407106612112068651403864036558911912665835891840}{B} \right)^2;$$

$$a^2t^2x^2 + dtx = \frac{A^2}{C^2} (117259729972169177544994691400724)^2,$$

$$b^2t^2x^2 + etx = \frac{A^2}{D^2} (177860189070384369048581732007492)^2,$$

$$c^2t^2x^2 + ftx = \frac{A^2}{E^2} (367665873141977567378862842430140)^2.$$

Since ax, bx, cx, ex fulfil conditions (2, 3, 4, 5, 6, 7, 8, 9, 10, 11); atx, btx, etx will satisfy the same conditions; because the two sets of numbers have the same *ratio* to each other.

3757. (Proposed by Miss LADD.)—In a quadrilateral ABCD, join the centres of gravity of the triangles ABC, ACD, ABD, BCD, and show that the resulting quadrilateral will be similar to ABCD, and that each side will be equal to one third of the corresponding side of ABCD.

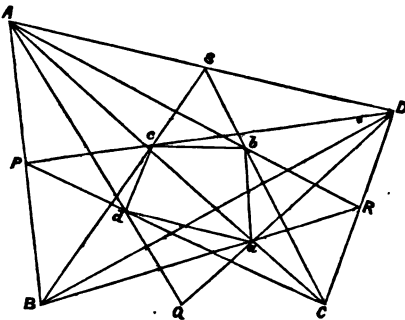
Solution by F. CAMPBELL.

Let ABCD be the quadrilateral; P, Q, R, S the middle points of its sides; and a, b, c, d the centres of gravity of the four triangles.

Then because $RA = 3Rb$, and $RB = 3Ra$; therefore ab is parallel to AB and equal to $\frac{1}{3}AB$.

Similarly all the sides of quadrilateral $abcd$ are parallel to the corresponding sides of ABCD, and equal to one-third of them respectively.

[A Solution by Quaternions is given on p. 96 of Vol. XVII. of the *Reprint*.]



3490. (Proposed by R. TUCKER, M.A.)— B', C', D' are the centres of the osculating circles at the points B, C, D of an ellipse, which co-intersect at a point A on the curve, and O is the centre of the circle through A, B, C, D. Prove (1) that B', C', D' have precisely the same relations on an ellipse, centre O, with axes parallel to the original axes, as B, C, D have on the given ellipse; and (2) that the locus of the centroid of the five points A, B, C, D, O is a sextic curve.

Solution by the PROPOSER.

The coordinates of A, B, C, D are respectively

$$ae^2 \cos^3 \delta, -\frac{a^2 e^2}{b} \sin^3 \delta; \quad ae^2 \cos^3 \frac{\delta}{3}, +\frac{a^2 e^2}{b} \sin^3 \frac{\delta}{3} \dots (1, 2);$$

$$-ae^2 \cos^3 \frac{\pi + \delta}{3}, -\frac{a^2 e^2}{b} \sin^3 \frac{\pi + \delta}{3}; \quad -ae^2 \cos^3 \frac{\pi - \delta}{3}, \frac{a^2 e^2}{b} \sin^3 \frac{\pi - \delta}{3} \dots (3, 4).$$

These may be written in the following forms:—

$$\frac{ae^2}{4} (\cos 3\delta + 3 \cos \delta), \quad -\frac{a^2 e^2}{4b} (3 \sin \delta - \sin 3\delta) \dots (1);$$

$$\frac{ae^2}{4} \left(\cos \delta + 3 \cos \frac{\delta}{3} \right), \quad +\frac{a^2 e^2}{4b} \left(3 \sin \frac{\delta}{3} - \sin \delta \right) \dots (2);$$

$$\frac{ae^2}{4} \left(\cos \delta - 3 \cos \frac{\pi + \delta}{3} \right), \quad -\frac{a^2 e^2}{4b} \left(\sin \delta + 3 \sin \frac{\pi + \delta}{3} \right) \dots (3);$$

$$\frac{ae^2}{4} \left(\cos \delta - 3 \cos \frac{\pi - \delta}{3} \right), \quad -\frac{a^2 e^2}{4b} \left(\sin \delta - 3 \sin \frac{\pi - \delta}{3} \right) \dots (4);$$

and the coordinates of O are (*Reprint*, Vol. XII., p. 38)

$$\frac{ae^2}{4} \cos \delta, \quad -\frac{a^2 e^2}{4b} \sin \delta;$$

whence we see that O is the centroid of (2), (3), (4). And if we transfer our origin to O , it is readily seen that B' , C' , D' are related in the same manner to the point where the central vector of A meets the similar ellipse through B' , C' , D' as the original four points are, the axes of the new ellipse being at right angles to the former and to them in the ratio of $3e^2 : 4$.

If (h, k) be the centroid of the five points, then

$$5h = ae^2 (\cos^3 \delta + \cos \delta), \quad 5k = -\frac{a^2 e^2}{b} (\sin^3 \delta + \sin \delta) \dots\dots (5, 6).$$

Assume $\cos^3 \delta + \cos \delta = \lambda$, $\sin^3 \delta + \sin \delta = \mu \dots\dots\dots (\alpha, \beta)$.

Squaring and adding, we get $\lambda^2 + \mu^2 = 4 - 7 \sin^2 \delta \cos^2 \delta = 4 - 7z^2$ suppose; also, multiplying the two equations (α) , (β) , we get

$$\lambda\mu = \cos^3 \delta \sin^3 \delta + 2 \sin \delta \cos \delta = z^3 + 2z.$$

Eliminating z , we obtain $343\lambda^2\mu^2 = \{18 - (\lambda^2 + \mu^2)\}^2 \{4 - (\lambda^2 + \mu^2)\}$;

or $(\lambda^2 + \mu^2)^3 - 40(\lambda^2 + \mu^2)^2 + 468(\lambda^2 + \mu^2) + 343\lambda^2\mu^2 = 1296$,

where $\lambda = \frac{5b}{ae^2}$, $\mu = -\frac{5kb}{a^2e^2}$; the equation to a sextic curve.

3123. (Proposed by A. MARTIN.)—Find the mean distance of the vertex of a right pyramid whose base is a rectangle (1) from all the points in its base, and (2) from all the points within the pyramid.

2379. (Proposed by S. WATSON.)—Find the mean distance of all the points in a rectangular parallelepiped from one of its angles; and hence show that the mean distance in a cube whose side is unity is

$$\frac{3^{\frac{1}{2}}}{4} - \frac{\pi}{24} + \log \frac{1+3^{\frac{1}{2}}}{2^{\frac{1}{2}}}.$$

Solution by STEPHEN WATSON.

3123. (1.) Let h = the height, and $2a$, $2b$ the sides of the base. If through the centre of the base lines be drawn parallel to the sides, they will divide the base into four equal portions, the average distance of the points in any one of which from the vertex will be the same as that for the whole base. Hence the average distance is

$$\begin{aligned} & \frac{1}{ab} \int_0^a dx \int_0^b dy (x^2 + y^2 + h^2)^{\frac{1}{2}} \\ &= \frac{1}{2ab} \int_0^a dx \left\{ b(b^2 + h^2 + x^2)^{\frac{1}{2}} + (h^2 + x^2) \log \frac{b + (b^2 + h^2 + x^2)^{\frac{1}{2}}}{(h^2 + x^2)^{\frac{1}{2}}} \right\} \\ &= \frac{1}{6ab} \left\{ 2abd + a(a^2 + 3h^2) \log \frac{b+d}{(a^2 + h^2)^{\frac{1}{2}}} + b(b^2 + 3h^2) \log \frac{a+d}{(b^2 + h^2)^{\frac{1}{2}}} \right. \\ & \quad \left. - 2bh^4 \int_0^a \frac{dx}{(h^2 + x^2)(b^2 + h^2 + x^2)^{\frac{1}{2}}} \right\}, \text{ where } d^2 = a^2 + b^2 + h^2 \dots\dots (A). \end{aligned}$$

Put $x = \frac{2h^2 - z}{2(b^2 - h^2 + z)^{\frac{1}{2}}}$; then the part requiring integration becomes

$$= 4b\dot{h}^4 \int_{2h^2}^{a^2(a^2+h^2-ad)} \frac{dz}{z^2 + 4b^2h^2} = 2h^2 \left\{ \tan^{-1} \left(\frac{a^2 + h^2 - ad}{bh} \right) - \tan^{-1} \left(\frac{h}{b} \right) \right\}$$

$= -2h^2 \tan^{-1} \left(\frac{ab}{hd} \right)$, and (A) becomes

$$= \frac{1}{6} \left\{ 2d + \frac{a^2 + 3h^2}{b} \log \frac{b+d}{(a^2+h^2)^{\frac{1}{2}}} + \frac{b^2 + 3h^2}{a} \log \frac{a+d}{(b^2+h^2)^{\frac{1}{2}}} - \frac{2h^2}{ab} \tan^{-1} \left(\frac{ab}{hd} \right) \right\}$$

..... (B).

3123. (2.) Take a plane parallel to the base at the distance z from the vertex; then, the ratio of a line to this plane, to the same line produced to the base, is $\frac{z}{h}$, and the ratio of the points in the plane, to those in the base, is $\frac{a^2}{h^2}$; and the chance that another parallel plane shall be indefinitely near to the supposed one, is $\frac{dz}{h}$; hence the mean distance in this case is

$$\frac{1}{h^4} \int_0^h z^3 (2) dz = \frac{1}{4} (B) \text{ (C).}$$

2379. If in (C), as interpreted by (B), we interchange first a and h , and then b and h , and add the two results and (C) together, we have the mean distance required in this question

$$= \frac{1}{4} d + \frac{a^2 + b^2}{6h} \log \frac{h+d}{(a^2+b^2)^{\frac{1}{2}}} + \frac{b^2 + h^2}{6a} \log \frac{a+d}{(b^2+h^2)^{\frac{1}{2}}} + \frac{a^2 + h^2}{6b} \log \frac{b+d}{(a^2+h^2)^{\frac{1}{2}}} \\ - \frac{1}{12} \left\{ \frac{a^3}{bh} \tan^{-1} \left(\frac{bh}{ad} \right) + \frac{b^3}{ah} \tan^{-1} \left(\frac{ah}{bd} \right) + \frac{h^3}{ab} \tan^{-1} \left(\frac{ab}{hd} \right) \right\} \text{ (D).}$$

In the case of a cube, $a=b=h$, and (D) becomes

$$= \frac{a}{4} \left(3^{\frac{1}{2}} + 4 \log \frac{1+3^{\frac{1}{2}}}{2^{\frac{1}{2}}} - \frac{\pi}{6} \right).$$

3770. (Proposed by T. MITCHELSON, B.A., F.C.P.)—If x_1, px_1 be the abscissas of two points on a parabola, and x_2 the abscissa of the point of intersection of the tangents at the first two points; prove that $x_2 = p^{\frac{1}{2}}x_1$.

Solution by J. H. SLADE; Q. W. KING; the PROPOSER; and others.

Let the equation to the parabola be $y^2 = 4ax$; then the ordinates of the points x_1 and px_1 are respectively $(4ax_1)^{\frac{1}{2}}$ and $(4apx_1)^{\frac{1}{2}}$; hence the equations to tangents at the ends of these ordinates will be

$$y(4ax_1)^{\frac{1}{2}} = 2a(x+x_1), \quad y(4apx_1)^{\frac{1}{2}} = 2a(x+px_1) \text{ (1, 2).}$$

Dividing (1) by (2), and writing x_2 for the x of the point of intersection,

we have $\frac{1}{p^{\frac{1}{2}}} = \frac{x_2 + x_1}{x_2 + px_1}, \quad \text{or} \quad x_2 = p^{\frac{1}{2}}x_1,$

3773. (Proposed by the Rev. Dr. BOOTH, F.R.S.)—1. Let

$$Ax^2 + A_1y^2 + 2Bxy + 2Cx + 2C_1y = 1$$

be the projective equation of a conic, and let a right-angle move along this curve, one side passing through the origin; then the other side will envelope a curve whose tangential equation is

$$A\xi^2 + A_1v^2 + 2B\xi v + 2(C\xi + C_1v)(\xi^2 + v^2) = (\xi^2 + v^2)^2.$$

2. Let $a\xi^2 + a_1v^2 + 2\beta\xi v + 2\gamma\xi + 2\gamma_1v = 1$ be the tangential equation of a conic, and from the origin let a perpendicular be drawn on a tangent to the conic; then the locus of the feet of the perpendiculars will be the curve, whose projective equation is

$$ax^2 + a_1y^2 + 2\beta xy + 2(\gamma x + \gamma_1 y)(x^2 + y^2) = (x^2 + y^2)^2.$$

Solution by J. DALE; A. B. EVANS, M.A.; and others.

Let O be the origin, AB any straight line, OP perpendicular to AB; x, y the projective coordinates of P; (ξ, v) the tangential coordinates of AB; then

$$\frac{x}{\xi} = \frac{y}{v} = \frac{(x^2 + y^2)^{\frac{1}{2}}}{(\xi^2 + v^2)^{\frac{1}{2}}} = \frac{x^2 + y^2}{\xi^2 + v^2} = \frac{1}{\xi^2 + v^2};$$

consequently, if P lies in a curve whose projective equation is $f(x, y) = 0$, the tangential equation to the envelope of AB will be

$$f\left(\frac{\xi}{\xi^2 + v^2}, \frac{v}{\xi^2 + v^2}\right) = 0;$$

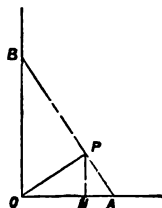
also, if AB touches a curve whose tangential is $\phi(\xi, v) = 0$, then P will lie in a curve whose projective equation is $\phi\left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2}\right)$.

[For example, the tangential equation to the four-cusped hypocycloid is

$$\xi^{-2} + v^{-2} = a^2;$$

therefore the locus of the foot of the perpendicular from the centre is

$$(x^2 + y^2)^3 = a^2 x^2 y^2.]$$



3815. (Proposed by J. J. WALKER, M.A.)—Show that the locus of feet of perpendiculars let fall from points in a given diameter of a quadric surface on the polar planes of those points is a rectangular hyperbola.

Solution by C. LNUESDORF; the PROPOSER; and others.

Any point on a diameter of $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$\frac{x'}{l} = \frac{y'}{m} = \frac{z'}{n} = r \text{ say } \dots\dots\dots (1).$$

Its polar plane is

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1 \dots\dots\dots (2).$$

Perpendicular on this is $\frac{a^2}{x'}(x-x') = \frac{b^2}{y'}(y-y') = \frac{c^2}{z'}(z-z') \dots\dots\dots (3).$

From (3) we have $\frac{mx-ly}{lm\left(\frac{1}{a^2}-\frac{1}{b^2}\right)} = \frac{ny-mz}{mn\left(\frac{1}{b^2}-\frac{1}{c^2}\right)},$

or $mn\left(\frac{1}{b^2}-\frac{1}{c^2}\right)x + nl\left(\frac{1}{c^2}-\frac{1}{a^2}\right)y + lm\left(\frac{1}{a^2}-\frac{1}{b^2}\right)z = 0 \dots\dots(a).$

Also, by (3), $r = \frac{1}{m-l}\left\{\frac{a^2x}{l}-\frac{b^2y}{m}\right\} = \frac{1}{\frac{xl}{a^2}+\frac{ym}{b^2}+\frac{zn}{c^2}};$ by (1) and (2);

therefore $\left(\frac{xl}{a^2}+\frac{ym}{b^2}+\frac{zn}{c^2}\right)\left(\frac{xm}{b^2}-\frac{yl}{a^2}\right) = \frac{lm(m-l)}{a^2b^2} \dots\dots\dots (\beta).$

Hence the locus is given by (a) and (β). But (β) is a hyperbolic cylinder, the sections of which by planes perpendicular to the line of intersection of its asymptotic planes are rectangular hyperbolas. But (a) cuts

the line $\frac{xl}{a^2}+\frac{ym}{b^2}+\frac{zn}{c^2}=0, \quad \frac{xm}{b^2}-\frac{yl}{a^2}=0$

at right angles. Hence the required locus is a rectangular hyperbola.

3405. (Proposed by Professor WOLSTENHOLME.)—If S, H be two fixed points at a distance $2a$, C the middle of SH, P a point such that the rect-angle SP . HP = c^2 ($< a^2$); show that the radius of curvature of the locus of P at any point P is $\frac{b^2c^2\sqrt{2}}{a^3-\beta^3}$, where $a^2 \equiv a^2 \cos 2\theta + b^2$, $\beta^2 \equiv a^2 \cos 2\theta - b^2$, $b^2 \equiv (a^4 - c^4)^{\frac{1}{2}}$, and $\theta = \angle PCS$. Hence prove that the locus of the points of minimum curvature, for different values of c , is the lemniscate $r^2 = a^2 \cos 2\theta$; and that the tangent at any point of minimum curvature passes through C.

Solution by the Rev. G. H. HOPKINS, M.A.; S. WATSON; G. S. CARR; and others.

The locus is represented by $c^4 = (a^2 + r^2 - 2ar \cos \theta)(a^2 + r^2 + 2ar \cos \theta)$. This equation takes the various forms

$$2a^2r^2 \cos 2\theta - r^4 = b^4, \quad r^2 = a^2 \cos 2\theta + a\beta, \quad \text{and} \quad r\sqrt{2} = a + \beta.$$

It is easily seen that $\frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2 = \frac{c^4}{r^2(a^2 \cos 2\theta - r^2)^2} = \frac{1}{\beta^2};$

therefore $p = \frac{r}{c^2}(a^2 \cos 2\theta - r^2)$, therefore $\frac{dp}{dr} = -\frac{a^2 \cos 2\theta + r^2}{c^2};$

and $r \frac{dr}{dp} = -\frac{c^2 r}{a^2 \cos 2\theta + r^2};$

therefore $\rho = \frac{c^2 r}{a^2 \cos 2\theta + r^2} = \frac{c^2(a + \beta)}{\sqrt{2}(2a^2 \cos 2\theta + a\beta)} = \frac{b^2 c^2 \sqrt{2}}{(a^2 + a\beta + \beta^2)(a - \beta)} = \frac{b^2 c^2 \sqrt{2}}{a^3 - \beta^3}.$

Now
$$\rho = \frac{c^2 r}{a^2 \cos 2\theta + r^2} = \frac{c^2 r}{\frac{b^4 + r^4}{2r^2} + r^2} = \frac{2c^2 r^3}{b^4 + 3r^4}.$$

That the curvature may be a maximum we must have $r^4 = b^4 = a^4 - c^4$.

Eliminating c from $2a^2 r^2 \cos 2\theta - r^4 = b^4$, there remains $r^2 = a^2 \cos 2\theta$.

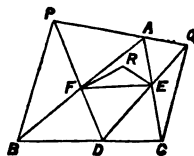
The perpendicular from the pole C upon the tangent vanishes when $r^2 = a^2 \cos 2\theta$, which is the case at all the points of minimum curvature, or the tangent at points of minimum curvature passes through the pole.

3838. (Proposed by Professor CROFTON, F.R.S.)—If the two principal stresses at any point of a plane lamina be equal and of contrary signs, show that the resultant stresses on the three sides of any triangular element of the lamina at that point meet on the nine-point circle of the triangle.

Solution by J. HOPKINSON, D.Sc.

Let ABC be the triangle; D, E, F the middle points of the sides; PAQ parallel to one axis of stress; BP and CQ parallel to the other. Then since the stresses are equal and opposite and perpendicular to PB and PA , the resultant stress on AB will be perpendicular to PF , say FR ; and that on AC perpendicular to EQ , say along ER .

And since the angles R and D are together equal to two right angles, therefore R is on the nine-point circle.



3785. (Proposed by A. STEIN.)—Prove that for all real values of a_1, a_2, \dots, a_n ,

$$\sum_{k=1}^{k=n} \frac{a_k^{n-2}}{(a_k - a_1) \dots (a_k - a_{k-1}) (a_k - a_{k+1}) \dots (a_k - a_n)} = 0.$$

Solution by MATTHEW COLLINS, B.A.

By decomposing $\frac{x^{n-2}}{(x-a)(x-b)(x-c)(x-d)(x-e) \dots}$ into partial fractions

$$= \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} + \frac{E}{x-e} \dots \text{to } n \text{ terms;}$$

we find $A = \frac{a^{n-2}}{(a-b)(a-c)(a-d) \dots}, B = \frac{b^{n-2}}{(b-a)(b-c)(b-d) \dots}, \&c.;$

also by clearing of fractions the assumed equation

$$\frac{x^{n-2}}{(x-a)(x-b)(x-c)(x-d)(x-e)\dots} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c} + \frac{D}{x-d} + \dots;$$

we find $x^{n-2} = A(x-b)(x-c)(x-d)(x-e) + B(x-a)(x-c)(x-d)(x-e) + C(x-a)(x-b)(x-d)(x-e) + \dots$

Hence, equating the coefficients of x^{n-1} at both sides, we get

$$A + B + C + D + E + \dots = 0,$$

and this will still be true if the numerators a^{n-2} , b^{n-2} ... be of the forms $P + Qa + Ra^2 \dots La^{n-2}$, $P + Qb + Rb^2 \dots Lb^{n-2}$, $P + Qc + Rc^2 \dots Lc^{n-2}$, &c.

3760. (Proposed by G. S. CARR.)—Find when the light thrown upon a round table from a luminous point over the centre is a maximum; also when a smaller concentric circle is excluded from the surface.

Solution by Professor TOWNSEND, F.R.S.

The quantity of light received by any area from a luminous point, supposing none of it to be lost by absorption on the way in its passage through the atmosphere, being evidently proportional to the solid angle subtended by the area at the point; in the case of a point on the axis of a circular disc, the point must consequently be at the centre of the disc, in which case the solid angle of subtense $= 2\pi$; and in the case of a point on the axis of a circular ring whose external and internal radii are a and b , its height above the plane of the ring may be readily determined as follows:—

Denoting by α and β the semi-angles of the two cones of revolution subtended by the outer and inner circles at the point, the solid angle in question is then $= 2\pi(\cos \beta - \cos \alpha)$, and the question is to find h so that $\cos \beta - \cos \alpha$, or $\frac{h}{(h^2 + b^2)^{\frac{1}{2}}} - \frac{h}{(h^2 + a^2)^{\frac{1}{2}}}$, shall be a maximum; which, by

application of the ordinary method, gives for h the value $\frac{a^{\frac{1}{2}}b^{\frac{1}{2}}}{(a^{\frac{1}{2}} + b^{\frac{1}{2}})^{\frac{1}{2}}}$, which accordingly is the height required.

3718. (Proposed by F. D. THOMSON, M.A.)—The locus of a point such that a tangent from it to a given conic is cut in a given ratio by a given straight line is a curve whose equation is of the form

$$pR^2UV + qRJZ + rJ^2LM = 0,$$

where p, q, r are constants; J is the line at infinity; U, V are the asymp-

totes of the given conic; L, M the tangents at the points where the given line R meets the conic; and Σ is a certain conic having asymptotes parallel to U and V . Trace the possible changes of form which the locus may assume.

Solution by the PROPOSER.

Take the diameters parallel and conjugate to R as axes of y and x , then $R \equiv x - a = 0$ is the equation to the given line, and if (x', y') be the point of contact corresponding to the point (xy) on the locus,

$$Ax'x' + By'y' = 1, \text{ where } Ax'^2 + By'^2 = 1 \dots\dots\dots (1, 2).$$

If the given ratio be $m : l$, we have $lx + mx' = (l + m)a \dots\dots\dots (3)$. Hence eliminating x', y' from (1), (2), (3), we obtain as the equation to the locus

$$By^2 \{ A [l(a-x) + ma]^2 - m^2 \} + \{ lAx(a-x) + m(Aax-1) \}^2 = 0 \dots\dots (4).$$

Hence two asymptotes parallel to R are given by the equation

$$A \{ l(a-x) + ma \}^2 = m^2 \dots\dots\dots (5),$$

which will be real if A is positive, or if the axis of x meets the conic in real points. There are double points which may be conjugate or impossible at points given by $y=0$, $lAx(a-x) + m(Aax-1) = 0$.

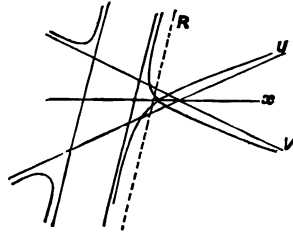
The equation (4) may also be written

$$A^2(a-x)^2 \{ Ax^2 + By^2 \} + 2lmA(a-x) \{ a(Ax^2 + By^2) - x \} + m^2 \{ By^2(Aa^2 - 1) + (Aax - 1)^2 \} = 0;$$

or, with the notation of the question, $pR^2UV + qRJ\Sigma + rJ^2LM = 0$.

It is evident that U and V are asymptotes to the locus, and it will be found that the parallel asymptotes pass through points of inflexion at infinity.

The locus will take various forms according as the given conic is an ellipse, an hyperbola, or a parabola, and according as R cuts, touches, or does not cut the conic. One of the most characteristic forms is given in the annexed figure; the others may be easily traced.



3800. (Proposed by W. Hogg.)—A particle falls towards a centre of force, of which the intensity varies inversely as the cube of the distance; to find the whole time of descent.

I. Solution by M. COLLINS, B.A.; A. B. EVANS, M.A.; A. MARTIN; and others.

Let the body fall from A to a centre of attraction S moving along APS , let $SA = a$, $SP = x$, $AP = s$, $m =$ the attraction of S on a material point at

the unit of distance from S; then, by the well established old principle $v dv = f ds$, we have in the present case $v dv = -\frac{m dx}{x^3}$, whose integral is $v^2 = m \left(\frac{1}{x^2} - \frac{1}{a^2} \right)$; showing that the velocity v is 0 (as it ought to be) when $x=a$ at A, and that $v=\infty$ when the body arrives at the centre of force S. This last equation gives $axv = m^{\frac{1}{2}}(a^2 - x^2)^{\frac{1}{2}}$, and as $v = -\frac{dx}{dt}$; therefore $\frac{-ax \cdot dx}{(a^2 - x^2)^{\frac{1}{2}}} = m^{\frac{1}{2}} dt$, whose integral is $m^{\frac{1}{2}} t = a(a^2 - x^2)^{\frac{1}{2}}$, which requires no correction as it gives $t=0$ when $x=a$. The required time when $x=0$, that is, when the attracted body coming from A arrives at the centre of force S, is $\frac{a^2}{m^{\frac{1}{2}}}$ seconds.

II. Solution by J. DALE; A. B. EVANS, M.A.; and others.

Taking the origin at the centre, the equation of motion is

$$\frac{d^2 x}{dt^2} = -\frac{\mu}{x^3}.$$

Multiplying by $2 \frac{dx}{dt}$ and integrating, we get

$$\left(\frac{dx}{dt} \right)^2 = v^2 = C + \frac{\mu}{x^2}.$$

Suppose the particle to start from rest at a distance a from the centre; then, when $x = a$, $v = 0$;

therefore $C = -\frac{\mu}{a^2}$, and $\left(\frac{dx}{dt} \right)^2 = \frac{\mu(a^2 - x^2)}{a^2 x^2}$;

therefore $\frac{dx}{dt} = -\frac{\mu^{\frac{1}{2}}}{a} \cdot \frac{(a^2 - x^2)^{\frac{1}{2}}}{x}$,

the negative sign being taken, because x diminishes as t increases;

therefore $dt = -\frac{a}{\mu^{\frac{1}{2}}} \cdot \frac{x}{(x^2 - a^2)^{\frac{1}{2}}} = \frac{a}{\mu^{\frac{1}{2}}} d \cdot (a^2 - x^2)^{\frac{1}{2}}$;

therefore $t = C + \frac{a(a^2 - x^2)^{\frac{1}{2}}}{\mu^{\frac{1}{2}}}$.

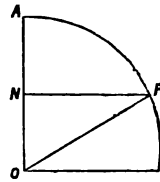
When $x = a$, $t = 0$, therefore $C = 0$; hence we have

$$t = \frac{a}{\mu^{\frac{1}{2}}} (a^2 - x^2)^{\frac{1}{2}} = \frac{a^2}{\mu^{\frac{1}{2}}}, \text{ when } x = a.$$

NOTE.—Let O be the centre of force, A the point where the motion begins; on OA describe a quadrant, and suppose the particle has fallen to N; then

$$\text{velocity at N} = \frac{\mu^{\frac{1}{2}}}{a} \tan \text{PON},$$

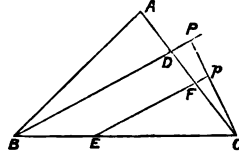
$$\text{time of falling} = \frac{a^2}{\mu^{\frac{1}{2}}} \sin \text{PON}.$$



3693. (Proposed by S. WARSON.)—A line is drawn at random so as to cut a given triangle, and two points are taken at random within the triangle. Find the probability that the two points lie on opposite sides of the line; and hence show that, if the triangle be equilateral, the probability becomes $\frac{1}{3} + \frac{1}{10} \log 3$.

Solution by the PROPOSER.

Let EF be the random line cutting BC, CA in E and F. Draw BD parallel to EF, and upon it let fall the perpendicular CP, cutting EF in P. Put $Cp = x$, $\angle CBD = \phi$, $\angle ABD = \phi_1$, triangle CEF = u , quadrilateral ABEF = u_1 , and triangle ABC = Δ . Also s, s_1, s_2, s_3 as usual. Then



$$u = \frac{1}{2} EF \cdot x = \frac{CE \cdot \sin C \cdot x}{2 \sin (C + \phi)} = \frac{\sin C \cdot x^2}{2 \sin \phi \sin (C + \phi)},$$

$$u_1 = \Delta - u.$$

Now the number of ways the two points can lie on opposite sides of EF is $2uu_1$; and the limits are x from 0 to $a \sin \phi$, ϕ from 0 to B. Hence we have

$$\int_0^B d\phi \int_0^{a \sin \phi} 2uu_1 dx = \Delta^2 \int_0^B d\phi \left\{ \frac{2a^2 \sin^2 \phi}{3b \sin (C + \phi)} - \frac{2a^3 \sin^3 \phi}{5b^2 \sin^2 (C + \phi)} \right\} \dots (1).$$

In like manner, when EF lies on the opposite side of BD, we have

$$\Delta^2 \int_0^B d\phi_1 \left\{ \frac{2c^2 \sin^2 \phi_1}{3b \sin (A + \phi_1)} - \frac{2c^3 \sin^3 \phi_1}{5b^2 \sin^2 (A + \phi_1)} \right\} \dots (2).$$

Put $C + \phi = \theta$, $A + \phi_1 = \theta_1$; then (1) + (2) reduces to

$$\begin{aligned} & \frac{2\Delta^2}{3b} \left[\int_0^{\pi-A} a^2 d\theta \left\{ \sin (\theta - 2C) + \frac{\sin^2 C}{\sin \theta} \right\} \right. \\ & \quad \left. + \int_A^{\pi-C} c^2 d\theta_1 \left\{ \sin (\theta_1 - 2A) + \frac{\sin^2 A}{\sin \theta_1} \right\} \right] \\ & - \frac{2\Delta^2}{5b^2} \left[\int_0^{\pi-A} a^3 d\theta \left\{ \sin (\theta - 3C) + \frac{3 \sin^2 C \cos C}{\sin \theta} - \frac{\sin^3 C \cos \theta}{\sin^2 \theta} \right\} \right. \\ & \quad \left. + \int_A^{\pi-C} c^3 d\theta_1 \left\{ \sin (\theta_1 - 3A) + \frac{3 \sin^2 A \cos A}{\sin \theta_1} - \frac{\sin^3 A \cos \theta_1}{\sin^2 \theta_1} \right\} \right] \\ & = \frac{4\Delta^2}{b^2} \sin \frac{1}{2} B \left[\frac{1}{2} b \left\{ a^2 \sin \frac{1}{2} (B - 2C) + c^2 \sin \frac{1}{2} (B - 2A) \right\} \right. \\ & \quad \left. - \frac{1}{2} \left\{ a^3 \sin \frac{1}{2} (B - 4C) + c^3 \sin \frac{1}{2} (B - 4A) \right\} \right] + \frac{8\Delta^4}{15b^3} \log \frac{s}{s_2} \\ & = \frac{4}{15} \Delta^2 s_2 s_1 (c + a) \left(\frac{2}{ca} - \frac{1}{b^2} \right) + \frac{8\Delta^4}{15b^3} \log \frac{s}{s_2} \dots (3). \end{aligned}$$

Similarly, when the line parallel to the random line passes through the angles C and A, and lies within the range of those angles, we have respectively

$$\frac{4}{15}\Delta^2 s_1 s_2 (a+b) \left(\frac{2}{ab} - \frac{1}{c^2} \right) + \frac{8\Delta^4}{15c^3} \log \frac{s}{s_3} \dots\dots\dots (4),$$

$$\frac{4}{15}\Delta^2 s_2 s_3 (b+c) \left(\frac{2}{bc} - \frac{1}{a^2} \right) + \frac{8\Delta^4}{15a^3} \log \frac{s}{s_1} \dots\dots\dots (5).$$

Again, the number of lines EF that can be drawn parallel to BD, while BD takes all positions within the angle B, is

$$\int_0^a d\phi \{ a \sin \phi + c \sin (B-\phi) \} = (c+a) (1-\cos B).$$

Similarly, for the others we have

$$(a+b)(1-\cos C) \quad \text{and} \quad (b+c)(1-\cos A);$$

hence the total number of lines and points is $2\Delta^2 s$, and the probability is

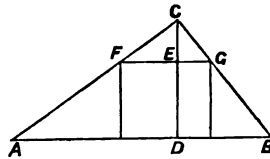
$$p = \frac{(3)+(4)+(5)}{2\Delta^2 s} = \frac{1}{30} \left\{ 4 + \frac{(b+c)(b-c)^2}{a^2 s} \right. \\ \left. + \frac{(c+a)(c-a)^2}{b^2 s} + \frac{(a+b)(a-b)^2}{c^2 s} + \frac{8\Delta^2}{s} \log \left(\frac{s}{s_1} \right)^{\frac{1}{s^2}} \left(\frac{s}{s_2} \right)^{\frac{1}{s^2}} \left(\frac{s}{s_3} \right)^{\frac{1}{s^2}} \right\}.$$

When $a=b=c$, then we have $p = \frac{2}{15} + \frac{1}{15} \log 3$.

3691. (Proposed by Dr. HART.)—In a plane triangle, find integers for the sides, diameter of the inscribed circle, and the side of the inscribed square, such that the sum of the diameter and side shall be a square.

Solution by the PROPOSER.

Let r = radius of inscribed circle,
 $AD = n(m^2-1)x$, $BD = m(n^2-1)x$,
 $CD = 2mnx$; then we have
 $AC = n(m^2+1)x$, $BC = m(n^2+1)x$,
 $AB = \{n(m^2-1) + m(n^2-1)\}x$
 $= (mn-1)(m+n)x$;



hence $(2m^2n + 2mn^2)x = 2mn(m+n)x$ = perimeter of the triangle,
 and $2mnr(m+n)x = 2\Delta = 2mn(mn-1)(m+n)x^2$,
 whence $2r = 2(mn-1)x$.

Again, by similar triangles, $AB : CD = FG$ (or DE) : CE ,
 that is, $(mn-1)(m+n)x : 2mnx = FG : 2mnx - FG$,

whence we have $FG = \frac{2mn(mn-1)(m+n)x}{2mn + (mn-1)(m+n)}$ = side of inscribed square;

hence $2(mn-1)x + \frac{2mn(mn-1)(m+n)x}{2mn+(mn-1)(m+n)} = \square = x^2$, and

$$x = 2(mn-1) + \frac{2mn(mn-1)(m+n)}{2mn+(mn-1)(m+n)} = \frac{2(mn-1)\{2mn+(2mn-1)(m+n)\}}{2mn+(mn-1)(m+n)}.$$

It is evident that, if the above expression for the sides of the triangle, diameter of inscribed circle, and side of inscribed square, be multiplied by $2mn+(mn-1)(m+n)$, we shall have integers, where m, n may be any numbers greater than 1. But if m be greater than 1 and n equal to 1, the segment BD will vanish, and we shall have the right-angled triangle ACD. Again, if $m=n>1$, the triangle ABC will be isosceles.

3636. (Proposed by the Rev. W. H. LAVERTY, M.A.)—P is a point on an ellipse of which the foci are S and S'. PS, PS' meet the curve again in Q and Q'. RQM is perpendicular to the major axis, RM being equal to the central abscissa of Q'. Show that the locus of R is a rectangular hyperbola.

I. Solution by N'IMPORTE.

If the eccentric angles of P, Q, Q' be θ, α, β respectively, the locus of R is given by

$$-x = a \cos \alpha, \quad -y = a \cos \beta \dots\dots (1).$$

Also, since PQ passes through S,

$$\tan \frac{1}{2}\theta \tan \frac{1}{2}\alpha + \frac{1+e}{1-e} = 0;$$

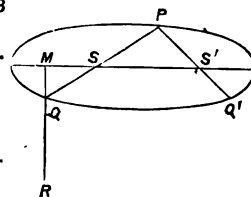
and since PQ' passes through S',

$$\tan \frac{1}{2}\theta \tan \frac{1}{2}\beta + \frac{1-e}{1+e} = 0;$$

therefore
$$\frac{\tan \frac{1}{2}\alpha}{\tan \frac{1}{2}\beta} = \left(\frac{1+e}{1-e}\right)^2.$$

But, by (1), $\tan \frac{1}{2}\alpha = \left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$, and $\tan \frac{1}{2}\beta = \left(\frac{a+y}{a-y}\right)^{\frac{1}{2}}$; therefore the

locus of R is $\frac{(a+x)(a-y)}{(a-x)(a+y)} = \left(\frac{1+e}{1-e}\right)^4$, a rectangular hyperbola.



II. Solution by STEPHEN WATSON; the PROPOSER; and others.

Denote the point P by (x', y') ; then the equations of PS, PS' are

$$y-y' = \frac{y'}{c+x'}(x-x'), \text{ and } y-y' = -\frac{y'}{c-x'}(x-x');$$

and eliminating y from each of these and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the resulting values

of x are a and $-\beta$, (a, β) being the coordinates of R , that is,

$$a = -\frac{(a^2 + c^2)x' + 2a^2c}{a^2 + c^2 + 2cx'}, \quad -\beta = -\frac{(a^2 + c^2)x' - 2a^2c}{a^2 + c^2 - 2cx'};$$

whence eliminating x' , we have

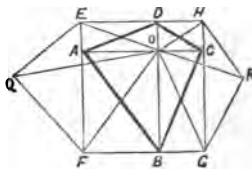
$$4ca\beta + \left(a^2 + c^2 + \frac{4a^2c^2}{a^2 + c^2}\right)(a + \beta) + 4a^2c = 0,$$

the equation of an hyperbola having its asymptotes parallel to the axes of the ellipse, and therefore rectangular.

3780. (Proposed by the Editor.)—If a, b, c, d be the lengths of the sides of a quadrilateral whose diagonals are equal and perpendicular to each other, show that its area is $\frac{1}{4}\{a^2 + c^2 + [4b^2d^2 - (a^2 - c^2)^2]^{\frac{1}{2}}\}$.

I. Solution by N'IMPORTE.

Through the vertices of the quadrilateral $ABCD$ draw lines parallel to the diagonals AC, BD ; then these lines will, by their intersections, form a square $EFGH$, the area of which is twice that of the quadrilateral. Moreover, if AC, BD intersect at O , each of the figures OE, OF, OG, OH is a rectangle; therefore the lines OE, OF, OG, OH are respectively equal to DA, AB, BC, CA , that



is to say, to a, b, c, d . Hence we have given the lines drawn to the angles of a square from a point within the figure, to determine its area. Now suppose the triangle OFG to revolve round the point F till FG coincides with FE , and FO takes the position FQ ; and, in like manner, suppose OE to revolve round H into the position RH . Then the triangle OGR is equal in area to OEQ ; for OG, GR are respectively equal to QE, EO , and the angles OGR, QEO are supplements of each other. Also $FO = FQ = b$, and OFQ is a right angle; therefore $OQ = b\sqrt{2}$; moreover $OE = a$, and $EQ = OG = c$; hence if Δ be the area of the triangle OEQ or OGR ,

$$\text{we have } 4\Delta^2 = 4a^2c^2 - (a^2 - 2b^2 + c^2)^2 = 4a^2c^2 - (b^2 - d^2)^2,$$

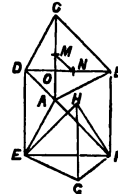
$$\text{since } b^2 - a^2 = BO^2 - OD^2 = c^2 - d^2 \text{ or } a^2 + c^2 = b^2 + d^2.$$

Putting now S for the required area of the quadrilateral $ABCD$, we have $S = \frac{1}{2}EFGH = \frac{1}{2}(OEQF + OGRH) = \frac{1}{2}(OFQ + OHR + 2OEQ)$
 $= \frac{1}{4}\{b^2 + d^2 + [4a^2c^2 - (b^2 - d^2)^2]^{\frac{1}{2}}\} = \frac{1}{4}\{a^2 + c^2 + [4b^2d^2 - (a^2 - c^2)^2]^{\frac{1}{2}}\}.$

[It is evident that the area of the quadrilateral may be expressed as a function of any three of the four given quantities a, b, c, d ; but the expression for the area given in the Question, as a function of all four of the quantities, is of a simpler form.]

II. *Solution by A. MARTIN; J. DALE; T. MITCHESON, B.A.; and others.*

Let ABCD be the quadrilateral, $AB=a$, $BC=b$, $CD=c$, $DA=d$, and $AC=BD$ (the equal diagonals) each $=x$. From B and D draw perpendiculars to AC, meeting AC in E and F (parallels to CD and CB respectively) in E and F. Join EF; then, by the question and construction, the figure BDEF is a square, $AF=BC=b$, $AE=CD=c$, and the area of the square is evidently double of that of the quadrilateral ABCD.



(Fig. 2.)

Let the angle $AFB = \theta$; then $AFE =$ complement of AFB ; hence we have

$$2bx \cos \theta = b^2 + x^2 - a^2, \text{ from the triangle } ABF,$$

$$2bx \sin \theta = b^2 + x^2 - c^2, \text{ from the triangle } AEF.$$

Square these two equations and take their sum; then we obtain

$$4b^2x^2 = (x^2 + b^2 - a^2)^2 + (x^2 + b^2 - c^2)^2;$$

$$\text{therefore } x^4 - (a^2 + c^2)x^2 = b^2(a^2 - b^2 + c^2) - \frac{1}{4}(a^4 + c^4) \dots\dots\dots (1).$$

But since the diagonals AC and BD are perpendicular to one another, we have $b^2 - a^2 (=CO^2 - OA^2) = c^2 - d^2$, or $a^2 - b^2 + c^2 = d^2$.

Substituting this value of $a^2 - b^2 + c^2$ in (1), and resolving, gives

$$S = \frac{1}{2}x^2 = \frac{1}{4}\{a^2 + c^2 + [4b^2d^2 - (a^2 - c^2)^2]^{\frac{1}{2}}\}.$$

If the quadrilateral is inscribable in a circle, the equal diagonals are equally distant from the centre, and the perpendiculars from the centre drawn to the middle points of the diagonals are equal, and form the adjacent sides of a square. This shows that the greater segments of the diagonals are equal to each other, and also the less segments are equal to each other. Hence, if m and n denote the greater and less segments of either diagonal, we have $2m^2 = a^2$ and $2n^2 = c^2$, and the area of the quadrilateral is

$$S = \frac{1}{2}(m+n)^2 = \frac{1}{4}(a+c)^2.$$

The quadrilateral in the question admits of the following simple construction. Make $AE=c$, draw EG at right angles to AE and take $EG=EA=c$. With centres A and G, and radii AF and GF, equal respectively to b and d , describe circular arcs intersecting in F; draw EF, then EF will be one of the diagonals of the quadrilateral, and GH, equal and perpendicular to EF, will be the other diagonal of the required quadrilateral EFGH.

III. *Solution by S. BILLS; E. BLACKWOOD; and others.*

Let ABCD (Fig. 2) be the quadrilateral, and suppose the equal diagonals AC, BD to intersect at right angles in O. Let $AB=a$, $BC=b$, $CD=c$, $DA=d$; and M, N being the middle points of the diagonals, put $AC=BD=2x$, $MO=y$, $NO=z$; then we have

$$(x-y)^2 + (x+z)^2 = a^2, \quad (x+y)^2 + (x+z)^2 = b^2 \dots\dots\dots (2, 3),$$

$$(x+y)^2 + (x-z)^2 = c^2, \quad (x-y)^2 + (x-z)^2 = d^2 \dots\dots\dots (4, 5).$$

From $\{(2) + (4)\}$, $\{(2) - (3)\}$, $\{(3) - (4)\}$, we obtain

$$2x^2 + y^2 + z^2 = \frac{1}{2}(a^2 + c^2), \quad y = \frac{b^2 - a^2}{4x}, \quad z = \frac{b^2 - c^2}{4x} \dots\dots\dots (6, 7, 8).$$

Substituting from (6) in (5), and reducing, we have

$$(2x)^2 - (a^2 + c^2)(2x)^2 = -\frac{1}{2} \{ (a^2 - b^2)^2 + (b^2 - c^2)^2 \} \\ = b^2(a^2 - b^2 + c^2) - \frac{1}{2}(a^4 + c^4),$$

which is the same as equation (1), since $2x$ here denotes each of the equal diagonals.

IV. Solution by MATTHEW COLLINS, B.A.

As the diagonals are perpendicular to each other $a^2 + c^2 = b^2 + d^2$; and therefore they must be always perpendicular to each other for any quadrilateral whose sides are $a = 25$, $b = 19$, $c = 31$, $d = 35$. Now if the quadrilateral be inscribable in a circle, the rectangle under its diagonals, or twice its area, is then $= ac + bd = 1440$, but in all other cases it is less than $ac + bd$; hence its maximum area is 720.

When the diagonals are equal to each other and each $= 2v$, the area $S = 2v^2$; let, then, x, y, z be the legs and hypotenuse of the right-angled triangle whose vertices are the middle points and the point of intersection of the two diagonals (M, N, O in Fig. 3); then we have

$$a^2 - b^2 = d^2 - c^2 = (v + x)^2 - (v - x)^2 = 4vx, \\ d^2 - a^2 = c^2 - b^2 = (v + y)^2 - (v - y)^2 = 4vy;$$

therefore

$$(d^2 - a^2)^2 + (d^2 - c^2)^2 = (4v)^2(x^2 + y^2) = 16v^2x^2 = 8Sx^2;$$

$$\text{and as } a^2 + b^2 + c^2 + d^2 = 2(2v)^2 + 4z^2 = 4(S + z^2), \quad (\text{Fig. 3.})$$

$$\text{therefore } S + z^2 = \frac{1}{2}(a^2 + c^2) = \frac{1}{2}(b^2 + d^2);$$

hence S and z^2 are the two roots of the quadratic

$$S^2 - \frac{1}{2}S(a^2 + c^2) + \frac{1}{4}\{(d^2 - a^2)^2 + (d^2 - c^2)^2\} = 0.$$

$$\text{But } (a^2 + c^2)^2 - 2(d^2 - a^2)^2 - 2(d^2 - c^2)^2 = 2a^2c^2 - a^4 - c^4 + 4d^2(a^2 + c^2 - b^2) \\ = 4b^2d^2 - (a^2 - c^2)^2;$$

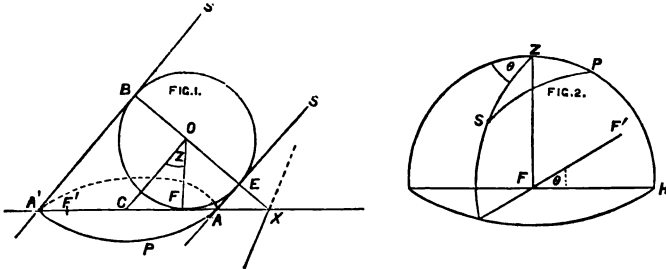
therefore the two roots are $\frac{1}{4}\{a^2 + c^2 \pm [4b^2d^2 - (a^2 - c^2)^2]^{\frac{1}{2}}\}$, the greater of which, viz. $\frac{1}{4}\{1586 + (1330^2 - 336^2)^{\frac{1}{2}}\} = 718.214$ is the required area S ; and $4z^2 = (1586 - 1286.858) = 299.142$, and we observe also that the sum of the squares of the two roots, or $S^2 + z^4$ is $= \frac{1}{4}(a^2c^2 + b^2d^2)$.

The quadrilateral is thus constructed geometrically. Take (Fig. 2) EA and EG perpendicular to each other, and each $= b = 19$; inflect $AF = a = 25$ and $GF = c = 31$ to meet in F ; then EF will be one of the diagonals, and GH perpendicular and equal to EF will obviously be the other diagonal of the required quadrilateral $EGFH$.

3380. (Proposed by Professor TOWNSEND, F.R.S.)—Assuming that one focus of the shadow of a sphere, standing on a horizontal plane exposed to the light of the Sun, is its point of contact with the plane; find the envelope of the corresponding directrix, and the locus of the remaining focus, for a given day and place.

I. *Solution by R. W. GENESE, B.A.*

If O be the centre of the sphere, S the sun at any time; then OS describes a right cone. The locus of the centre C of the shadow is therefore a conic section, and the locus of the variable focus a similar curve. If F be the fixed focus, X the foot of the corresponding directrix, $FC \cdot FX =$ square of semi-minor axis; and this is constant, for the shadow is a section of a right cylinder circumscribing the sphere. Thus the envelope of the directrix is the reciprocal polar of the locus of C , and is therefore also a conic section.

II. *Solution by F. D. THOMSON, M.A.*

Let Fig. (1) represent a vertical section of the sphere passing through the sun, supposed to be at an infinite distance, and let APA' be the shadow cast upon the horizontal plane, F the point of contact.

Produce the diameter BE to meet $A'A$ produced in X . Then it may be shown, as in the corresponding case in the cone, that F is a focus, and X is the foot of the corresponding directrix of the ellipse APA' .

Take C the centre and F' the other focus, and let a be the radius of the sphere, z the zenith distance of the sun. Then $FF' = 2FC = 2a \tan z$, and $FX = a \cot z$; therefore $FF' \cdot FX = 2a^2$, and therefore the envelope of the directrix is the polar reciprocal of the locus of F' , with respect to the point F .

Let θ be the azimuth, λ the co-latitude, δ the north polar distance of S (Fig. 2); then writing r for FF' , $r = 2a \tan z$, where

$$\cos \delta = \cos z \cos \gamma - \sin z \sin \lambda \cos \theta;$$

therefore, eliminating z , $(r^2 + 4a^2) \cos^2 \delta = (2a \cos \lambda - r \sin \lambda \cos \theta)^2$;

or, writing x, y for $r \cos \theta, r \sin \theta$

$$(x^2 + y^2 + 4a^2) \cos^2 \delta = (2a \cos \lambda - x \sin \lambda)^2.$$

The equation to the locus, which is therefore an ellipse, parabola, or hyperbola according as $\delta < = > \frac{1}{2}\pi - \lambda$. From this we derive the envelope of the directrix by reciprocation.

III. *Solution by the PROPOSER.*

The centre of the variable shadow, and the directrix corresponding to its fixed focus, being, at any instant, the intersections with the horizontal plane of the line and plane through the centre of the sphere parallel and

perpendicular respectively to the direction of the sun at that instant; the locus of the former and the envelope of the latter, during the interval between sunrise and sunset, are consequently the intersections with the horizontal plane of the two reciprocal cones of revolution determined by the variable line and plane in question in the course of the day; and therefore, &c.

3776. (Proposed by Professor WOLSTENHOLME, M.A.)—1. Show that the "feet-perpendicular" lines of any triangle inscribed in a circle are all asymptotes of rectangular hyperbolas passing through the corners of the triangle.

2. If S be the focus of the parabola inscribed in a triangle ABC ; Aa , Bb , Cc the perpendiculars of the triangle; S' a point on the nine-point circle such that S , S' are at the ends of parallel and opposite radii of the circles ABC , abc ; show that the tangent at the vertex of the former parabola will be the axis of a parabola inscribed in abc , and having its focus at S' .

I. Solution by the PROPOSER.

Since any such rectangular hyperbola is self-conjugate to the triangle whose corners are the feet of the perpendiculars of the triangle, each such asymptote is an axis of a parabola inscribed in this latter triangle. (*Book of Math. Problems*, Quest. 790.) Hence the tangent at the vertex of a parabola inscribed in a triangle is also the axis of a parabola inscribed in the triangle whose corners are the feet of the perpendiculars of the former triangle. Hence from Mr. Ferrers' discovery that the envelope of the feet-perpendiculars is a three-cusped hypocycloid touching the nine-points circle, it follows that the envelope of the axis of a parabola inscribed in a given triangle is a three-cusped hypocycloid, the inscribed circle of which is the circumscribed circle of a triangle, which I discovered to be the case soon after Mr. Ferrers' investigation was published, mine being, of course, suggested by his. We see also that the envelope of the asymptotes of a rectangular hyperbola circumscribing a given triangle is a three-cusped hypocycloid touching in three points the nine-point circle; and that of the asymptotes of a rectangular hyperbola self-conjugate to a given triangle is one touching in three points the circumscribed circle.

II. Solution by JAMES DALE.

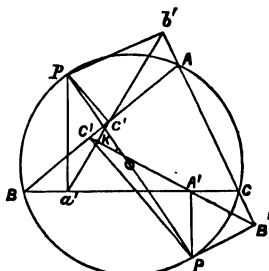
1. If S , s be points on the circumscribing circle diametrically opposite to each other, then the pedals corresponding to these points intersect at right angles in the nine-point circle; and if x_1y_1 , x_2y_2 , x_3y_3 be the perpendiculars from A , B , C respectively, then $x_1y_1 = x_2y_2 = x_3y_3 = SX \cdot sx$; where X , x are the perpendiculars from S , s on the corresponding pedals; therefore A , B , C lie on a rectangular hyperbola having a pair of pedals for asymptotes.

2. If H be the ortho-centre of ABC , then the middle point S' of SH lies on the nine-point circle, and also on the pedal of S with respect to ABC ;

hence S and S' are the extremities of parallel radii of the circles ABC , abc ; and as the triangle abc is similar to ABC but inverted in position, the pedal of S' with respect to abc is at right angles to the pedal of S with respect to ABC ; hence, if S' and the former pedal be respectively focus and vertical tangent of a parabola inscribed in abc , the latter pedal will be the axis.

III. Solution by MATTHEW COLLINS.

Let O be the centre, and POp any diameter of the circle about the triangle ABC , upon the sides of which the perpendiculars from P and p give the two straight lines $A'B'C'$ and $a'b'c'$. Now when a diameter of a circle is projected on any chord, the ends of the chord are equally distant from the ends of the projection; therefore $CB' = Ab'$, $CA' = Ba'$, and $BC' = Ac'$; and since the parts of any secant intercepted between a hyperbola and its asymptotes are equal to each other; therefore, if $A'B'C'$ and $a'b'c'$ (which are obviously perpendicular to each other) be the asymptotes of an hyperbola passing through A , it must also pass through B and C , since $Ab' = B'C$ and $CA' = Ba'$; therefore, &c.



3849. (Proposed by R. TUCKER, M.A.)—The circle through the focus of a parabola which touches the curve at P cuts it also in Q, R ; it is known that QR passes through a fixed point (O) on the axis: find the locus of the intersection of the tangent at P with QR , and the area of the portion of it comprised between O and its asymptote. Show that the perpendicular from P on QR cuts it in a parabola, vertex O , and touches a semicubical parabola, whose vertex is the foot of the given directrix; also find the locus of the centre of the circle and the area of its oval. The centroid of PQR lies on the axis.

Solution by C. LEUDESORF.

Taking $y^2 = 4ax$ as the equation to the parabola, and $px + qy + r = 0$ as the equation to QR , any conic touching the curve at $P \left(\frac{a}{m^2}, \frac{2a}{m} \right)$ and passing through Q and R is denoted by $y^2 - 4ax = \left(y - mx - \frac{a}{m} \right) (px + qy + r)$.

If this denote a circle, $p - qm = 0$ and $q = 1 - pm$;

whence $p = \frac{m}{m^2 + 1}$, and $q = \frac{1}{m^2 + 1}$.

And if the focus lie on it, $r = \frac{4am}{m^2 + 1} - pa = \frac{3am}{m^2 + 1}$;

therefore the equation to QR is $mx + y + 3am = 0$ (1);
showing that it cuts the axis in the fixed point $(-3a, 0)$.

The equation to the tangent at P is $y = mx + \frac{a}{m}$ (2),

and eliminating m between (1) and (2), the locus of their intersection is

$$y^2 = -a \frac{(x+3a)^2}{2x+3a}.$$

The area of this curve between $x = -3a$ and its asymptote $x = -\frac{3}{2}a$ is .

$$\begin{aligned} 2a^{\frac{1}{2}} \int_{-3a}^{-\frac{3}{2}a} \frac{x+3a}{\{(2x+3a)\}^{\frac{3}{2}}} dx &= a^{\frac{1}{2}} \int_0^{(3a)^{\frac{1}{2}}} (x^2-3a) dr \text{ if } -x^2 = 2x+3a, \\ &= 2a^2\sqrt{3}. \end{aligned}$$

The equation to the perpendicular from P on (1) is $m^2y - 2am = m^2x - a$... (3);
and eliminating m between (1) and (3), we have

$$[y^2 + (x+3a)^2][y^2 - a(x+3a)] = 0.$$

Thus the locus is a parabola, having its vertex at O. The envelope of (3), found by the usual method, is $27ay^2 = 4(x+2a)^3$, a semicubical parabola, vertex $(-2a, 0)$. The coordinates of the centre of the circle

$$(m^2+1)(y^2-4ax) = \left(y - mx - \frac{a}{m}\right)(mx+y+3am)$$

$$\text{are } x = \frac{a(m^2+3)}{2m^2}, \quad y = \frac{a(3m^2-1)}{2m^3};$$

whence, eliminating m , the locus is $27ay^2 = (2x-a)(x-5a)^2$,

$$\text{and the area of the oval} = \int_{-5a}^{-2a} \frac{2}{3(3a)^{\frac{1}{2}}} (x-5a)(2x-a)^{\frac{1}{2}} dx = \frac{18\sqrt{3}}{5} a^2.$$

3602. (Proposed by G. S. CARR.)—If a particle starting from rest at the point (a, b) be attracted to the axes of X and Y by forces $-vy$, $-\mu x$ respectively, show that the equation to the path is

$$\nu^{\frac{1}{2}} \cos^{-1} \frac{x}{a} = \mu^{\frac{1}{2}} \cos^{-1} \frac{y}{b},$$

which includes all the curves made by Mr. H. ARRY's pendulum, and described by him in the July and September numbers of *Nature*.

Solution by the PROPOSER.

The equations of motion towards the axes of y and x respectively are

$$x = A \cos(\mu^{\frac{1}{2}}t + B), \quad v = \mu^{\frac{1}{2}}A \sin(\mu^{\frac{1}{2}}t + B);$$

$$y = A' \cos(\nu^{\frac{1}{2}}t + B'), \quad v' = \nu^{\frac{1}{2}}A' \sin(\nu^{\frac{1}{2}}t + B');$$

when v, v' are the velocities at the time t .

When $x=a$ and $y=b$, we have $v=0$, $v'=0$, $t=0$; therefore $A=a$, $A'=b$, $B=0$, $B'=0$; and the equations for x and y become

$$x = a \cos(\mu^{\frac{1}{2}}t), \quad y = b \cos(\nu^{\frac{1}{2}}t).$$

Eliminate t , and we obtain the result given in the question for the equation to the orbit.

3782. (Proposed by Professor CROFTON, F.R.S.)—Find the distribution of a weight placed on a table among the four legs of the table, supposing the slab inflexible.

Solution by Professor TOWNSEND, F.R.S.

Denoting by A, B, C, D the four points of support of the slab by the four legs of the table, by I the intersection of the two diagonals AC and BD of the quadrilateral ABCD, which to fix the ideas is supposed to be of the ordinary non-reentrant form; by O the vertical projection on the horizontal plane ABCD of the centre of gravity of the entire supported weight W, the point O being, to fix the ideas, supposed to be within the area of the triangle AIB; by a, b, c, d the areas of the four triangles BCD, CDA, DAB, ABC; by m and n those of the two AOC and BOD; by P, Q, R, S the four required pressures; and by $\alpha, \beta, \gamma, \delta$ the four compressions produced by them in the four legs, to which on the ordinarily received principles they are proportional, and which, in consequence of the assumed rigidity of the slab, are connected geometrically by the linear relation $a\alpha - b\beta + c\gamma - d\delta = 0$; we have then, combining the consequent geometrical with the three ordinary statical relations, for the determination of P, Q, R, S, the four linear equations

$$aP - bQ + cR - dS = 0, \quad P + Q + R + S = W \dots\dots\dots (1, 2),$$

$$cP - aR = nW, \quad dQ - bS = mW \dots\dots\dots (3, 4);$$

whose solutions by the ordinary method give consequently

$$P = \left[\frac{a(b^2 + d^2) + a(b-d)m + \{(b^2 + d^2) + c(b+d)\}}{(a^2 + b^2 + c^2 + d^2)(b+d)} \right] W \dots\dots (5),$$

$$Q = \left[\frac{b(a^2 + c^2) + b(a-c)n + \{(a^2 + c^2) + d(a+c)\}m}{(a^2 + b^2 + c^2 + d^2)(a+c)} \right] W \dots\dots (6),$$

$$R = \left[\frac{c(b^2 + d^2) + c(b-d)m - \{(b^2 + d^2) + a(b+d)\}n}{(a^2 + b^2 + c^2 + d^2)(b+d)} \right] W \dots\dots (7),$$

$$S = \left[\frac{d(a^2 + c^2) + d(a-c)n - \{(a^2 + c^2) + d(a+c)\}m}{(a^2 + b^2 + c^2 + d^2)(a+c)} \right] W \dots\dots (8);$$

formulae which of course become considerably simplified when, as is usually the case, the quadrilateral ABCD is a parallelogram; the four triangles a, b, c, d being then all equal, and each = half the area of the parallelogram.

3803. (Proposed by Rev. Dr. BOOTH, F.R.S.)—If $f(x, y) \equiv F = 0$ be the projective equation of a plane curve, show that the tangential equation of its evolute may be found by eliminating x and y from the equations

$$\xi = \frac{dF}{dy} \div \left(\frac{dF}{dy} x - \frac{dF}{dx} y \right), \quad v = -\frac{dF}{dx} \div \left(\frac{dF}{dy} x - \frac{dF}{dx} y \right), \quad f(x, y) = 0;$$

and show therefrom that the tangential equation to the evolute of the ellipse is $a^2v^2 + b^2\xi^2 = (a^2 - b^2)^2 \xi^2 v^2$.

3829. (Proposed by Rev. Dr. BOOTH, F.R.S.)—A right angle revolves round the origin of coordinates of a curve whose tangential equation is $\phi(\xi, v) \equiv \Phi = 0$, and one side of the angle meets a tangent to the curve in its point of contact; show that the other side of the right angle meets this tangent in a point whose locus may be determined in projective coordinates by the three equations

$$x = \frac{d\Phi}{dv} \div \left(\frac{d\Phi}{dv} \xi - \frac{d\Phi}{d\xi} v \right), \quad y = -\frac{d\Phi}{d\xi} \div \left(\frac{d\Phi}{dv} \xi - \frac{d\Phi}{d\xi} v \right), \quad \phi(\xi, v) = 0;$$

and show therefrom that the locus for the ellipse is

$$a^2b^2(a^2x^2 + b^2y^2) = (a^2 - b^2)^2 x^2 y^2.$$

Solution by C. LEUBSDORF.

The tangential equation to the evolute, being the condition that $\xi x + v y = 1$ should touch the evolute, is found by identifying this equation with that of the normal to $f(x, y) = 0$, viz. $(x' - x) \frac{dF}{dy} - (y' - y) \frac{dF}{dx} = 0$.

Hence $\frac{\xi}{\frac{dF}{dy}} = \frac{v}{-\frac{dF}{dx}} = \frac{1}{x \frac{dF}{dy} - y \frac{dF}{dx}}$ and $f(x, y) = 0$ (A).

Reciprocating with regard to the origin of coordinates, and writing, therefore, tangential coordinates for projective, and *vice versa*, we have the theorem in Question 3829, that the locus of a point Q on the tangent at P such that $\angle POQ = \frac{1}{2}\pi$ is given by equations (A), where (ξ, v) are now projective, and x, y tangential coordinates.

(1) For the ellipse the projective equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{therefore} \quad \frac{dF}{dx} = \frac{2x}{a^2}, \quad \frac{dF}{dy} = \frac{2y}{b^2};$$

$$\text{therefore} \quad \xi = \frac{a^2}{x(a^2 - b^2)}, \quad v = \frac{-b^2}{y(a^2 - b^2)};$$

and therefore $\frac{a^2}{\xi^2} + \frac{b^2}{v^2} = (a^2 - b^2)^2$ is the tangential equation to the evolute.

(2) The tangential equation of the ellipse is $a^2\xi^2 + b^2v^2 = 1$; therefore

$$\frac{dF}{d\xi} = 2a^2\xi, \quad \frac{dF}{dv} = 2b^2v;$$

$$\text{therefore} \quad \xi = \frac{-b^2}{x(a^2 - b^2)}, \quad v = \frac{-a^2}{y(a^2 - b^2)};$$

and therefore $\frac{a^2}{y^2} + \frac{b^2}{x^2} = \frac{(a^2 - b^2)^2}{a^2b^2}$ is the locus of Q for the ellipse.

3812. (Proposed by the Editor.)—A person borrows £10000: how long will it take him to clear off the debt by paying off £100 at the end of the first year, £200 at the end of the second year, £300 at the end of the third, and so on, allowing 5 per cent. compound interest on the whole?

I. Solution by H. Hoskins; and others.

Let P = the principal or sum borrowed, $R = (1 + r)$ = the amount of £1 in one year at r per cent., a = the sum paid at end of first year, and n = the number of years required; then we have

$$PR^n = a \left\{ \begin{array}{c} 1 + R + R^2 + R^3 + \dots + R^{n-2} + R^{n-1} \\ 1 + R + R^2 + R^3 + \dots + R^{n-2} \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ 1 + R \\ 1 \end{array} \right\}$$

$$= a \{ R^{n-1} + 2R^{n-2} + 3R^{n-3} + \dots + (n-1)R + n \}.$$

Let $S = \{ R^{n-1} + 2R^{n-2} + 3R^{n-3} + \dots + (n-1)R + n \}$;
therefore $RS = \{ R^n + 2R^{n-1} + 3R^{n-2} + \dots + nR \}$,
whence $(R-1)S = R(R^{n-1} + R^{n-2} + \dots + R + 1) - n = R \frac{R^n - 1}{R - 1} - n$,

and $S = \frac{1}{R-1} \left\{ \frac{R(R^n - 1)}{R-1} - n \right\}$;

therefore $PR^n = aS = a \frac{1}{R-1} \left\{ \frac{R(R^n - 1)}{R-1} - n \right\}$,

whence $P = \frac{a}{R-1} \left\{ \frac{R(1 - R^{-n})}{R-1} - nR^{-n} \right\}$.

Now $P = 10000$, $a = 100$, $R = 1.05$; therefore

$$5 = 21(1 - R^{-n}) - nR^{-n}, \text{ and } R^n = \frac{1}{14}(21 + n).$$

Hence n is found by trial to lie between 18 and 19 years.

II. Solution by the Proposer.

Putting S_n for the sum due after the n th payment, that is to say, at the end of the n th year, we shall have

$$\begin{aligned} S_1 &= 10000 \left(\frac{3}{2}\right) - 100; \\ S_2 &= 10000 \left(\frac{3}{2}\right)^2 - 100 \left(\frac{3}{2}\right) - 200; \\ S_3 &= 10000 \left(\frac{3}{2}\right)^3 - 100 \left(\frac{3}{2}\right)^2 - 200 \left(\frac{3}{2}\right) - 300; \\ &\vdots \\ S_n &= 10000 \left(\frac{3}{2}\right)^n - 100 \left(\frac{3}{2}\right)^{n-1} - 200 \left(\frac{3}{2}\right)^{n-2} - \dots - n(100) \\ &= 10000 \left(\frac{3}{2}\right)^n - 100 \left(\frac{3}{2}\right)^{n-1} \left\{ 1 + 2 \left(\frac{2}{3}\right) + 3 \left(\frac{2}{3}\right)^2 + \dots + n \left(\frac{2}{3}\right)^{n-1} \right\} \\ &= 100000 \left(\frac{3}{2}\right)^n - 100 \left(\frac{3}{2}\right)^{n-1} \left\{ \frac{1 - (n+1) \left(\frac{2}{3}\right)^n + n \left(\frac{2}{3}\right)^{n+1}}{(1 - \frac{2}{3})^2} \right\} \end{aligned}$$

(see the method of summation given in the first solution); hence

$$S_n = 10000 \left(\frac{2}{3}\right)^n - 42000 \left\{ \left(\frac{2}{3}\right)^n - (n+1) + n \left(\frac{2}{3}\right) \right\} \\ = 2000 \left\{ n + 21 - 16 \left(\frac{2}{3}\right)^n \right\}.$$

This is a general expression for the sum due after the n th payment; hence the number (n) of payments required to clear off the debt will be given by the equation $n + 21 = 16 \left(\frac{2}{3}\right)^n$.

This equation may be most readily solved by Double Position, and we easily find that n lies between 18 and 19. The sum due after the 18th payment is $S_{18} = 988.1856$; also $S_{19} = -862.4064$; hence, to clear off the debt, he would have to pay (not 1900 but) £1037. 12s. nearly, at the last payment, supposing it to be made at the end of the 19th year.

III. *Solution by A. B. EVANS, M.A.; J. DALE; and others.*

Let x be the number of years required; then

$$\begin{aligned} \text{Present worth of £100 due 1 year hence} &= 100 \left(\frac{2}{3}\right), \\ \text{,, ,, £200 due 2 years hence} &= 200 \left(\frac{2}{3}\right)^2, \\ \text{,, ,, £100}x \text{ due } x \text{ years hence} &= 100x \left(\frac{2}{3}\right)^x; \end{aligned}$$

and the sum of all these present values is

$$100 \left(\frac{2}{3}\right) \left\{ 1 + 2 \left(\frac{2}{3}\right) + 3 \left(\frac{2}{3}\right)^2 + 4 \left(\frac{2}{3}\right)^3 + \dots + x \left(\frac{2}{3}\right)^{x-1} \right\}.$$

Multiplying and dividing this sum by $(1 - \frac{2}{3})^2$, it becomes

$$100 \left(\frac{2}{3}\right) \frac{1 - (x+1) \left(\frac{2}{3}\right) + x \left(\frac{2}{3}\right)^{x+1}}{(1 - \frac{2}{3})^2}.$$

But this sum = £10000, by the question; therefore

$$10000 = 42000 \left\{ 1 - (x+1) \left(\frac{2}{3}\right)^x + x \left(\frac{2}{3}\right)^{x+1} \right\},$$

therefore $x + 21 = 16 (1.05)^x = 16 (1 + .05)^x$

$$= 16 \left\{ 1 + \frac{x}{1} (.05) + \frac{x^2 - x}{1.2} (.05)^2 + \frac{x^3 - 3x^2 + 2}{1.2.3} (.05)^3 \right. \\ \left. + \frac{x^4 - 6x^3 + 11x^2 - 6x}{1.2.3.4} (.05)^4, \text{ \&c.} \right\}$$

Reducing and solving for x we find $x = 18.26$ nearly, or 19 payments are required of which the last is less than £1900.

3631. (Proposed by C. W. MERRIFIELD, F.R.S.)—Show that all cones satisfy the differential equations

$$\frac{\alpha\gamma - \beta^2}{r} = \frac{\alpha\delta - \beta\gamma}{2s} = \frac{\beta\delta - \gamma^2}{t}, \text{ where } \alpha = \frac{d^2z}{dx^2}, \beta = \frac{d^2z}{dx^2 dy}, \text{ \&c.};$$

and that, when the vertex moves off to infinity, the numerators vanish simultaneously.

Solution by J. HOPKINSON, D.Sc.

One of the properties of the cone is that its normals are parallel to the

generators of another cone; hence p and q are connected by a relation, say

$$f(pq) = 0 \dots\dots\dots (1);$$

hence $\frac{df}{dp} r + \frac{df}{dq} s = 0$, $\frac{df}{dp} s + \frac{df}{dq} t = 0$; therefore $rt - s^2 = 0 \dots\dots (2)$.

Differentiating (2) with respect to x and y ,

$$at + \gamma r - 2s\beta = 0, \quad \beta t + \delta r - 2s\beta = 0;$$

whence $\frac{\alpha\gamma - \beta^2}{r} = \frac{\alpha\delta - \beta\gamma}{2s} = \frac{\beta\delta - \gamma^2}{t}$ is true of all cones.

If the cone be a cylinder, relation (1) becomes $lp + mq + n = 0$.

Hence we have, successively, $lr + ms = 0$, $ls + mt = 0$;

$$la + m\beta = 0, \quad l\beta + m\gamma = 0, \quad l\gamma + m\delta = 0;$$

$$\alpha\gamma = \beta^2, \quad \alpha\delta = \gamma\beta, \quad \beta\delta = \gamma^2.$$

3591. (Proposed by Professor CAYLEY.)—If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation

$$\alpha \cdot \text{PAB} + \beta \cdot \text{PBC} + \gamma \cdot \text{PCD} + \delta \cdot \text{PDA} = 0$$

which connects the areas of the triangles PAB, &c.

Solution by MATTHEW COLLINS, B.A.

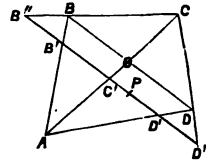
In the figure we have, evidently, $\frac{\Delta \text{PAB}}{\Delta \text{OAB}} = \frac{\text{PB}}{\text{OB}}$, $\frac{\Delta \text{PCD}}{\Delta \text{OCD}} = \frac{\text{PD}''}{\text{OD}}$, $\frac{\Delta \text{PBC}}{\Delta \text{OBC}} = \frac{\text{PB}''}{\text{OB}}$, $\frac{\Delta \text{PAD}}{\Delta \text{OAD}} = \frac{\text{PD}'}{\text{OD}}$.

Now $\frac{\text{OB}}{\text{OD}} = \frac{\text{C'B}''}{\text{C'D}''} = \frac{\text{C'B}'}{\text{C'D}'} = \frac{\text{B'B}''}{\text{D'D}''}$;

therefore $\frac{\text{B'B}''}{\text{OB}} = \frac{\text{D'D}''}{\text{OD}}$,

that is $\frac{\text{PB}'' - \text{PB}'}{\text{OB}} = \frac{\text{PD}'' - \text{PD}'}{\text{OD}}$; therefore $\frac{\text{PB}'}{\text{OB}} + \frac{\text{PD}''}{\text{OD}} = \frac{\text{PB}''}{\text{OB}} + \frac{\text{PD}'}{\text{OD}}$;

therefore, &c.



3735. (Proposed by ARTEMAS MARTIN.)—A ball is fired at random from a gun, with a given velocity; find the average of its range.

Solution by the PROPOSER.

If θ be the elevation of the gun, and h the height due to the velocity, the range is $2h \sin 2\theta$. The number of presentations of the gun at an ele-

vation θ is proportional to $\cos \theta$, supposing every elevation to be equally probable; hence the mean range is

$$\frac{\int_0^{1\pi} 2h \sin 2\theta \times \cos \theta d\theta}{\int_0^{1\pi} \cos \theta d\theta} = 2h \int_0^{1\pi} \sin 2\theta \cos \theta d\theta = \frac{4h}{3} = \frac{2v^2}{3g}.$$

3817. (Proposed by M. COLLINS, B.A.)—A body will describe a circle by a force $= v^2 r^{-1}$ directed towards its centre. Hence prove that if a circle be drawn round any other interior point C as centre of attraction, the force of attraction towards C must vary as $PC \div$ (the cube of the distance of P from the polar of C); P being any position of the point moving in the circumference; and show that this last property is true for all conic sections.

Solution by J. J. WALKER, M.A.

1. Let CT be the perpendicular on the tangent at P, and O the centre of the circle. If f be the force along PC, its component along PO must be equal to $\frac{v^2}{OP}$; that is, $f \times \frac{CT}{CP} \propto \frac{1}{CT \cdot OP}$, or $f \propto \frac{CP}{CT^3 \cdot OP}$. But in the circle CT is to the perpendicular from P on the polar of C as CO to radius; therefore, &c.

2. Let K be the centre of a conic, and KL, KM the semi-diameters conjugate to the directions KC, KP. If O be the centre of curvature at P, from above, $f \propto \frac{CP}{CT^3 \cdot OP} \propto \frac{CP}{CT^3 \cdot KM^3}$. But it is readily shown that the perpendicular from P on the polar of C, a fixed point, is proportional to $CT \cdot KM$; thus is proved the last elegant extension of the theorem, which is due to the late Sir Wm. Rowan Hamilton.

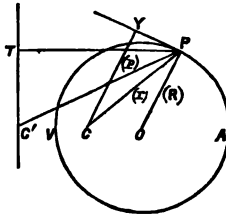
II. *Solution by the PROPOSER.*

The force urging P towards C multiplied by $\cos CPO$ is equal to the force urging the moving body P towards the centre $O = \frac{v^2}{R} = \frac{h^2}{Rp^3}$;

hence the force towards C

$$= \frac{h^2}{Rp^3 \cos CPO} = \frac{h^2 r}{Rp^3}.$$

Now let $OC \cdot OC' = R^2 = OP^2$; then $C'T$ perpendicular to OC' and to PT is polar of C; then since $OC' : OP = OP : OC$, the angle $OPC = OC'P$; that is, $\angle PCY = C'PT$; hence the triangle CPY is similar to $PC'T$; therefore $CY : PT = PC : PC'$, which is constant, since C and C' are inverse points relative to the circle. Hence the force acting on P towards C, and causing P to describe the circle O, varies as $PC : PT^3$.



3846. (Proposed by W. H. H. HUDSON, M.A.)—The *Times* of Thursday, Jan. 4, 1872, is numbered 27264. Supposing the paper to have been published every week-day without intermission, and numbered consecutively; give the day of the week, month, and year when the first number was published.

Solution by M. COLLINS; H. MURPHY; and others.

It took $(27264 \div 6 =)$ 4544 weeks exactly to issue the 27264 newspapers; and as the last day of all these weeks was Thursday, the first day of each must have been Friday. Now these 4544 weeks are more than 87 years, amongst which (the first and last being 1785 and 1871) there were 20 leap years, since, by the Gregorian correction, 1800 was a common year; and $87 \times 365 + 20 = 31775$ days are less than 4544 weeks by 33 days. But if the last of 33 consecutive days be the 4th of January, the first of them must be the 3rd of December of the preceding year; hence the first number of the *Times* must have been issued on Friday, December 3rd, 1784.

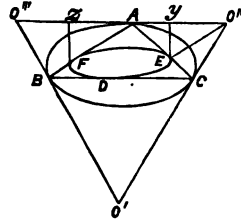
3845. (Proposed by C. TAYLOR, M.A.)—A triangle inscribed in an ellipse envelopes a confocal. Ellipses are described, each passing through a vertex, and having its foci at the adjacent points of contact. Show that the sum of the areas of their minor auxiliary circles is constant.

Solution by F. D. THOMSON, M.A.

Let D, E, F be the points where the sides of the triangle touch the inner conic; then (SALMON'S *Conics*, p. 196) it is known that the tangents at A and C intersect on the normal at E. Hence E is the point where the escribed circle touches the side AC. Hence

$$\begin{aligned} AE = BD = s - c, \quad AF = CD = s - b, \\ CE = BF = s - a, \end{aligned}$$

with the usual notation. Therefore, if EY, FZ be drawn perpendicular to O'' , O''' ,



$$EY \cdot FZ = b_1^2 \text{ suppose} = AE \cdot AF \cos^2 \frac{1}{2}A = (s-c)(s-b) \frac{s(s-a)}{bc} = \frac{4S^2}{bc},$$

where S is the area of ABC; therefore, similarly, we obtain

$$b_1^2 + b_2^2 + b_3^2 = \frac{4S^2}{abc} (a + b + c) = \frac{8S^2 s}{abc}.$$

Now it is known, by GRAVES' Theorem, that $s = \text{constant}$; therefore we have to show that $\frac{S^2}{abc}$ is constant.

Now, referring both conics to the triangle ABC, their equations are seen to be $yx + zx + xy = 0$, $\sin \frac{1}{2}A \cdot x^2 + \sin \frac{1}{2}B \cdot y^2 + \sin \frac{1}{2}C \cdot z^2 = 0 \dots (1, 2)$. Hence, forming the invariants, we have

$$\begin{aligned}\Theta &= -(\sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B + \sin^2 \frac{1}{2}C)^2, \\ \Theta' &= 4 \sin^2 \frac{1}{2}A \sin^2 \frac{1}{2}B \sin^2 \frac{1}{2}C (\sin^2 \frac{1}{2}A + \&c.), \\ \Delta &= 2, \quad \Delta' = -4 \sin^4 \frac{1}{2}A \sin^4 \frac{1}{2}B \sin^4 \frac{1}{2}C;\end{aligned}$$

therefore $\left(-\frac{\Delta'}{2\Delta}\right)^{\frac{1}{2}} = \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$

$$= \left\{ \frac{(s-b)(s-c)}{bc} \frac{(s-c)(s-a)}{ca} \frac{(s-a)(s-b)}{ab} \right\}^{\frac{1}{2}} = \frac{(s-a)(s-b)(s-c)}{abc} = \frac{4S^2}{s \cdot abc}.$$

But s is constant, therefore $\frac{S^2}{abc}$ is constant.

It will be found that the ratios of the other invariants give no new relation.

COR. 1.—The ratio of the triangle $O'O''O'''$ to ABC is constant, for this ratio may be shown to be $\frac{s \cdot abc}{4S^2}$.

COR. 2.—The ratio of the triangle DEF to ABC is constant, for

$$2\Delta DEF = 2S - (s-a)(s-b) \sin C - \&c.;$$

therefore $\frac{\Delta DEF}{S} = 1 - \frac{1}{2S} \left\{ (s-a)(s-b) \sin C + \&c. \right\}$

$$= 1 - \frac{1}{2S} (ab \sin C \sin^2 \frac{1}{2}C + \&c.) = 1 - (\sin^2 \frac{1}{2}C + \sin^2 \frac{1}{2}A + \sin^2 \frac{1}{2}B)$$

$$= \cos^2 \frac{1}{2}C - \sin^2 \frac{1}{2}A - \sin^2 \frac{1}{2}B - 2 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C$$

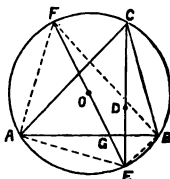
$$= \frac{8S^2}{s \cdot abc} = \text{constant}.$$

This agrees with Mr. WOLSTENHOLME'S theorem, Question 3165 (*Reprint*, Vol. XV., p. 43), so far as the *constancy* of the ratio.

3855. (Proposed by Professor WOLSTENHOLME.)—Three points A, B, C move on the arc of a fixed circle so that the centre of perpendiculars of the triangle ABC is a fixed point; prove that their velocities are to one another as $\sin 2A : \sin 2B : \sin 2C$.

Solution by Professor TOWNSEND, F.R.S.

Let ABC be any position of the variable triangle; D that of its fixed orthocentre; O the centre of the fixed circle on which its three vertices move; E the intersection with the circle of the perpendicular CD from any one of its vertices C upon the opposite side AB ; F the point on the circle diametrically opposite to E ; and G the intersection with AB of the diameter EF ; then AB touching in every position an ellipse of which D and O are the foci, and G being evidently its point of contact with the ellipse, and



therefore its intersection with its consecutive position, we have at once

$$\frac{\text{velocity of } A}{\text{velocity of } B} = \frac{AG}{BG} = \frac{AE \cdot AF}{BE \cdot BF} = \frac{\cos A \sin A}{\cos B \sin B} = \frac{\sin 2A}{\sin 2B};$$

and therefore, &c.

NOTE.—Throughout the deformation of the variable triangle, the angular velocity of any side being evidently half the sum of the angular velocities of its extremities round the centre of the circle, it follows at once from the above that angular velocity of a : angular velocity of b : angular velocity of $c = a \cos (B-C) : b \cos (C-A) : c \cos (A-B)$, and also that $\frac{\text{angular velocity of } a}{\text{angular velocity of } A} : \frac{\text{angular velocity of } b}{\text{angular velocity of } B} : \frac{\text{angular velocity of } c}{\text{angular velocity of } C} = \frac{\cos (B-C)}{\cos A} : \frac{\cos (C-A)}{\cos B} : \frac{\cos (A-B)}{\cos C}$.

3778. (Proposed by T. COTTERILL, M.A.)—1. If 1, 2, 3, 4, 5 are the successive sides of a plane pentagon, show that the tangent at the point (2, 4) to the conic through the five intersections of its non-successive lines cuts the line 3, in the same point as the line through the points (1, 2) and (4, 5).

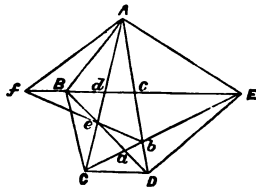
2. Prove that a cubic through these five points, touching the conic at the point (2, 4) and passing through the point (1, 5) cuts the lines 2, 3, 4 again in collinear points, and that two such cubics intersect again in two points collinear with (1, 5).

Solution by JAMES DALE.

1. Let $abcde$ be the pentagon formed by the lines 1, 2, 3, 4, 5; and $ABCDE$ the pentagon formed by the points (2, 4), (3, 5), &c. Then regarding A, A, C, E, B, D as an inscribed hexagon, the opposite sides AA, EB ; AC, BD ; CE, DA intersect respectively in three points which are collinear, which proves (1).

2. The straight line BE meets the cubic in the point which is *opposite* to the four points A, A, C, D (Salmon's *Higher Curves*, 133), and this point is collinear with the points where AC, AD meet the cubic, which proves the first part of (2).

Again, if two cubics have six points in common A, A, B, C, D, E lying on a conic, the three remaining points of intersection lie on a straight line (*Higher Curves*, 24), which proves the second part of (2).



3697. (Proposed by C. TAYLOR, M.A.)—If O be the orthocentre of a triangle ABC , the straight lines joining the mid-points of AO, BC ; BO, CA ; CO, AB make with AB, BC, CA angles complementary to C, A, B .

Solution by R. W. GENESER, B.A.

Let S be the centre of the circumscribing circle; D the mid-point of BC; X of OA; then $DS = \frac{1}{2}AO : AX$; therefore SA is parallel to DX. But SA makes with AB an angle equal to the complement of C; hence the theorem.

3874. (Proposed by Sir JAMES COCKLE, F.R.S.)—Writing, for brevity, $\frac{dy}{dx} = p$, and $\frac{d^2y}{dx^2} = \frac{dp}{dx} = r$, let there be given $nx^2r = (y - xp)^2 \dots (1)$.

Now BOOLE (*Differential Equations*, 2nd ed., p. 218) in effect states that

$$y = cx \dots \dots \dots (2)$$

is the singular solution of (1). Test this statement.

Solution by the PROPOSER.

1. We may write (1) thus: $nx^2 \frac{d}{dx} (xp - y) = (xp - y)^2$,

whereof the reciprocal of $x^2 (xp - y)^2$ is an integrating factor. Integrating,

we have $\frac{n}{xp - y} = \frac{1}{x} + C$; whence $xp - y = n \left(\frac{x}{1 + Cx} \right)$,

or $\frac{d}{dx} \left(\frac{y}{x} \right) = \frac{n}{x(1 + Cx)} = n \left(\frac{1}{x} - \frac{C}{1 + Cx} \right) \dots \dots \dots (3)$.

Integrating (3) and reducing, $y = nx \log \left(\frac{Cx}{1 + Cx} \right) \dots \dots \dots (4)$,

a result which may be put under the form $y = nx \log \frac{x}{A + Bx} \dots \dots \dots (b)$,

given by BOOLE. Let $A = 0$, and $\log \frac{1}{B} = \frac{c}{n}$; then (b) becomes (2), and I infer that (2) is a particular integral, and not the singular Solution of (1). See BOOLE, *ib.*, p. 229, Art. 9.

3696. (Proposed by G. S. CARR.)—A uniform rod of length a , elastic only to flexure, has its ends fastened to a string of length $b < a$. Determine the nature of the curve.

3809. (Proposed by Dr. BALL.)—Show that the tension of the string of a prismatic bow is $\frac{\pi KI}{L^2} \left(1 + \frac{\pi^2 D^2}{8L^2} \right)$, where K is the coefficient of elasticity, I the moment of inertia of the section of the bow, L the length of the bow, and D the distance between the centre of the string and the centre of the bow. Also explain why the tension does not vanish when $D = 0$.

3893. (Proposed by Dr. BALL.)—Prove (α) that the equation to the neutral axis of the bow is the result of eliminating ϕ between

$$x = \left(\frac{KI}{T}\right)^{\frac{1}{2}} \{2E(c, \phi) - F(c, \phi)\} \text{ and } y = D(1 - \cos \phi), \text{ where } c^2 = \frac{D \cdot T}{4KI}$$

(β.) That the entire length of the bow is $2 \left(\frac{KI}{T}\right)^{\frac{1}{2}} F(c)$.

(γ.) That the length of the bow and string together is $4 \left(\frac{KI}{T}\right)^{\frac{1}{2}} E(c)$.

Solution by Dr. BALL.

$$\text{The equation of the bow is } \frac{\frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}} + \frac{T}{KI} (y - D) = 0,$$

the origin being at the centre of the bow, the axis of y perpendicular to the string, and x parallel thereto.

$$\text{Integrating, we have } 1 - \frac{1}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}} + \frac{T}{2KI} \{(y - D)^2 - D^2\} = 0;$$

$$\text{make } \frac{dy}{dx} = \tan \theta \text{ and } D - y = D \cos \phi;$$

then we find $\sin \frac{1}{2}\theta = c \sin \phi$,

whence we readily deduce the result (α).

The length of the bow follows from the same substitutions, and is

$$2 \left(\frac{KI}{T}\right)^{\frac{1}{2}} \int_0^{\pi} \frac{1}{(1 - c^2 \sin^2 \phi)^{\frac{1}{2}}} d\phi = 2 \left(\frac{KI}{T}\right)^{\frac{1}{2}} F(c).$$

$$\text{The length of the string is } 2 \left(\frac{KI}{T}\right)^{\frac{1}{2}} \{2E(c) - F(c)\};$$

whence the length of bow and string together is found to be as in (γ).

If the deflection be small, powers > 2 of c may be neglected, and we have

$$L = 2 \left(\frac{KI}{T}\right)^{\frac{1}{2}} \int_0^{\pi} \left(1 + \frac{D^2 T}{8EI} \sin^2 \phi\right) d\phi,$$

$$\text{or } L = \pi \left(\frac{KI}{T}\right)^{\frac{1}{2}} \left(1 + \frac{\pi^2 D^2}{16L^2}\right);$$

$$\text{whence } T = \frac{\pi^2 KI}{L^2} \left(1 + \frac{\pi^2 D^2}{8L^2}\right).$$

If $D = 0$, the tension, instead of vanishing, attains the definite value $\frac{\pi^2 KI}{L^2}$. It is, however, to be observed, that the bending of the bow depends

upon the *bending moment*: for the deflection to vanish it is therefore only necessary for the bending moment to vanish, but if the deflection be zero, the bending moment of which the deflection is a factor, necessarily vanishes, whatever be the value of the tension. In this the bow may be contrasted with a beam supported at each end and laden at the centre. The bending moment in the latter case is the product of half the load by half the length of the beam, and this moment cannot vanish unless the load vanish.

Conceive a bow whose string is held vertically to rest with its lower end upon a horizontal plane. Let the string be attached at the upper end of the bow and pass through a hole in the lower end, and then through a hole in the plane. By attaching weights to the string the relation between the tension and the deflection can be studied. Any weight less than the critical value $\frac{\pi^2 KI}{L^2}$ will not bend the bow, and if the bow be forcibly bent it will straighten itself again in spite of the weight. If, however, a weight exceeding the critical value be attached, the bow will not return to the rectilinear condition when displaced therefrom, but will retain a deflection found by solving for D from the equation

$$\text{Weight} = \frac{\pi^2 KI}{L^2} \left(1 + \frac{\pi^2 D^2}{8L^2} \right).$$

It may be remarked that this result contains the solution of the problem of the deflection of a pillar with rounded ends. For this purpose we replace the symbol for the tension by a symbol expressing the load pressing upon the end of the pillar.

[Dr. BALL remarks that he owes Mr. CARR an apology for having inadvertently, in Question 3893, proposed as an independent problem what is really only a Solution of (at least one case of) Mr. CARR's Question 3696, and should have been contributed as such.]

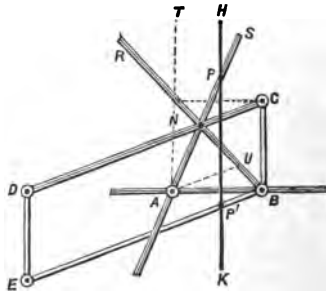
3875. (Proposed by Professor CAYLEY.)—Given the constant a and the variables x, y , to construct mechanically $\frac{a^2 - x^2}{y}$; or what is the same thing, given the fixed points A, B , and the moving point P , to mechanically connect therewith a point P' such that PP' shall be always at right angles to AB , and the point P' in the circle APB .

Solution by G. S. CARR.

Let there be a system of jointed rods as shown in the figure, in which BC is fixed at right angles to and equal to AB , and BR is fixed bisecting the angle ABC ; $BCDE$ a parallelogram with hinge joints; AS a rod working upon a hinge joint at A .

Let slots be cut along the axis of each of the rods BR, CD, AS , and let them be connected by a pin N , passing through all three.

Let a rod HK be made to slide at right angles to AB and carry a pin P passing through the slot in the rod AS . Then the required point P' will be the intersection of HK and BE .



P may be made to move at pleasure, and P' will always be on the circumference of the circle circumscribing ABP.

The proof is briefly: Draw AU, AT parallel to CD, HK. Then

$$\angle ABP' = \angle BAU = \angle TAP = \angle APP'.$$

[The preceding Question is not mathematical, but is interesting as relating to the mechanical description of curves. Prof. CAYLEY states that the problem presented itself to him in seeking for a mechanism for the description of cubic curves.]

3856. (Proposed by the EDITOR.)—Three men bought a grinding-stone 48 inches in diameter, 8 inches thick at the centre, and 12 inches thick at the circumference; each paying one-third of the expense. What part of the radius must each grind down for his share?

Solution by A. RENSCHAW; A. B. EVANS, M.A.; and others.

Let $x = ON$, the radius of the portion of the grinding-stone ABCDEF remaining for the third person; then the volume of the grinding stone, or of any concentric part of it, is the excess of a cylinder above two equal and opposite cones; and the height DI of one of these cones is found from the proportion $48 : 2 = 2x : DI$, whence $DI = \frac{1}{3}x$.

Now the volume of the whole stone is evidently

$$24^2\pi \cdot 12 - 24^2\pi \cdot \frac{4}{3} = 32 \cdot 24 \cdot 8\pi;$$

and the volume of the third person's share is

$$\pi x^2 \left(\frac{1}{3}x + 8\right) - \pi x^2 \left(\frac{1}{3}x\right) = \pi x^2 \left(\frac{1}{3}x + 8\right).$$

Equating this to one-third of the whole volume and reducing, we get $x^3 + 72x^2 = 18432$ (1);

and equating it to two-thirds of the whole volume we get

$$x^3 + 72x^2 = 36864$$
 (2).

From (1) we find $x = 14.58991$ inches, and from (2), $x = 20.01568$ inches; therefore the part of the radius each must grind down is

(1) 3.96864; (2) 5.42577; (3) 14.58991 inches.

[If, to obtain a *general* solution, we put $OG = a$, $GA = b$, $OD = c$, so that a is the radius of the stone, and b, c half the thickness at the edge and centre respectively, we have

$$\begin{aligned} \text{vol. stone ADEFCB} &= \text{vol. cylinder AEFB} - 2 \text{ vol. cone ADE} \\ &= \frac{1}{3}\pi (2AB + CD) OG^2 = \frac{2}{3}\pi a^2 (2b + c), \end{aligned}$$

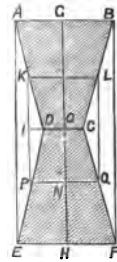
which is the same as that of a cylindrical stone of equal diameter, and of the *uniform* thickness of $\frac{1}{3}(AB + CD + EF)$, that is, in the question, of the uniform thickness of $10\frac{2}{3}$ inches.

Similarly, if KDPQCL be *any* part (n suppose, where n is a proper fraction) of the stone, and we put $ON = x$, $\tan DAE = \frac{b-c}{a} = m$, we have

$$\text{vol. KDPQCL} = \frac{1}{3}\pi (2KL + CD) ON^2 = \frac{2}{3}\pi x^2 (2mx + 3c);$$

hence x will be given by the equation

$$x^2 (2mx + 3c) = na^2 (2b + c)$$
 (a).



In the case proposed in the question, we have $a=24$, $b=6$, $c=4$, $m=\frac{1}{12}$, hence, making $n=\frac{1}{4}$ and $\frac{3}{4}$ respectively, we may obtain from (a) the equations (1) and (2) of the foregoing solution.]

3756. (Proposed by M. COLLINS, B.A.)—If a cardioid whose cusp is C be described by a moving body attracted towards an interior point C' , prove that this attraction towards C' varies as $\frac{PC \cdot PC'}{(CP \cdot C'P)^3}$; PP' touching the curve at P and meeting $C'P$ drawn through C' parallel to CP .

Solution by N'IMPORTE.

By Prop. vii., Cor. 3, Sect. ii. of Newton's *Principia*, if f be the force under the action of which to the cusp C the body would describe the cardioid we have

$$\text{force } (f') \text{ to } C' = f \frac{CP \cdot CP^2}{C'P^3}$$

Hence the question reduces itself to proving that $f \propto CP^{-4}$.

Supposing the cardioid generated by rolling the circle (O') on the equal circle (O); then if P be the point which traces out the curve, and N the point of contact at any moment of the two circles, $NP (=NC)$ is three-fourths of the radius of curvature at P ; therefore the chord of curvature through the centre of force C is equal to $\frac{3}{4}CP$. Hence, if CY , CZ be perpendicular and parallel to the tangent at P , we have

$$\frac{1}{f} \propto CY^2 \cdot CP \propto PZ^2 \cdot CP.$$

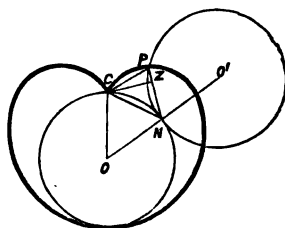
$$\text{Now } PZ = CP \cos CPN, \quad CP = 2CN \cos NCP,$$

$$CN = 2a \sin \frac{1}{2}(CON) = 2a \cos NCP, \quad \text{if } ON = a;$$

therefore $CP \propto \cos^2 NCP$, and $PZ \propto CP \cos NCP \propto CP^{\frac{3}{2}}$;

therefore $\frac{1}{f} \propto CP^4$; therefore $f' \propto CP^{-4} \cdot \frac{CP \cdot CP^2}{C'P^3} \propto \frac{PC \cdot PC'}{(PC \cdot PC')^3}$.

[N.B.—It will be seen that the figure of the cardioid is incorrectly engraved; since, obviously, the curve should extend to a distance equal to the diameter of the circle (O) below its lowest point. This error, however, in no wise affects the proof.]

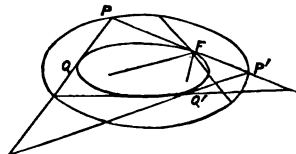


3970. (Proposed by Professor CROFTON, F.R.S.)—An ellipse, one of whose foci F is fixed, has double contact with a fixed ellipse E : show that its other focus describes an ellipse confocal with E and passing through F .

Solution by Professor WOLSTENHOLME.

Consider an ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and its focal curve $z = 0$, $\frac{x^2}{a^2 - c^2} + \frac{y^2}{b^2 - c^2} = 1$; the normal to the ellipsoid at any point is an axis of the cone whose vertex is the point and base the focal; or if F, F' be two fixed points on the focal, and P any point on the surface, the normal at which meets F, F' ; then FP, FP' will make equal angles with any tangent line through P . But in order that the normal at P may meet FF' , P must lie on a plane perpendicular to the plane of the focal, and since $FP, F'P$ make equal angles with the tangent to this curve throughout its length, $FP + F'P$ is constant throughout. Hence with F, F' as foci can be described a prolate spheroid having continuous contact with the ellipsoid along the curve which is the locus of P . If we take now the section of this prolate spheroid and ellipsoid by the plane of the focal, we see that if we take any two points on one of two given confocal ellipses, we can with those two points as foci describe an ellipse having double contact with the other given ellipse; or, which is the same thing, if with one fixed point F as focus we describe an ellipse having double contact with a given ellipse E , the second focus will lie on the ellipse through F confocal with E .

If F lie within the ellipse E , and PP' be the chord of E which touches at F the confocal through F ; $PQ, P'Q'$ tangents to the confocal; then for real contact the second focus F' must lie between Q and Q' . In the same way, by taking any two points on the focal hyperbola of the ellipsoid as foci, we may describe a prolate hyperboloid of revolution having continuous contact with the ellipsoid; so that when F is without the ellipse E , hyperbolas can be described with focus F having double contact with E , and the second focus F' will lie on the hyperbola through F confocal with E ; and for real contact on a certain arc of that hyperbola. (See *Book of Math. Problems*, Quest. 1181.)



NOTE.—If the tangents at F, F' meet in O , and OP, OP' be drawn to touch the confocal conic, P, P' will be the points where the ellipse with foci F, F' has its two contacts with the confocal, so that when O lies on the confocal the contact becomes of the third order, a property which I saw separately proved somewhere not long ago. I rather think Professor CROFTON'S property is proved in some of Mr. C. TAYLOR'S researches into the properties of confocal conics; it occurred to me as a consequence of the properties of the focal curves of quadrics, when I was concerned in those matters, much in the way I have given in my proof.

3887. (Proposed by F. D. THOMSON, M.A.)—Show that the equation to the cone of the second order whose sides are the lines 1, 2, 3, 4, 5 terminating at the origin, may be written

$$(014)(035)(245)(123) + (035)(013)(214)(245) + (025)(013)(435)(214) = 0,$$

where (014) $\equiv \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \end{vmatrix}$ &c., (x_1, y_1, z_1) being a point in the line 1, &c.

Quaternion Solution by R. F. SCOTT.

Let $\alpha, \beta, \gamma, \delta, \epsilon$ be the vectors 1, 2, 3, 4, 5, and let ρ be a variable vector given by the equation

$$S\alpha\beta\delta \cdot S\rho\gamma\epsilon \cdot S\beta\delta\epsilon \cdot S\alpha\beta\gamma + S\rho\gamma\epsilon \cdot S\alpha\beta\gamma \cdot S\beta\alpha\delta \cdot S\beta\delta\epsilon \\ + S\rho\beta\epsilon \cdot S\rho\alpha\gamma \cdot S\delta\gamma\epsilon \cdot S\beta\alpha\delta = 0 \dots\dots\dots (1)$$

Then this is the equation of a cone whose vertex is the origin, for the equation is not altered by writing $x\rho$ for ρ . Also it is a quadric cone, since ρ occurs twice in each term. Moreover the vectors $\alpha, \beta, \gamma, \delta, \epsilon$ lie on the cone, since the equation is satisfied if any one of them be put for ρ .

Now since
$$S\alpha\beta\delta = - \begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_4 & y_4 & z_4 \end{vmatrix} \text{ \&c.}$$

(1) is in fact Mr. THOMSON'S equation.

II. Solution by the PROPOSER.

It is obvious that the lines 1, 3, 5 lie in the surface, and if we write 2 for 0 throughout, we obtain the identical equation

$$(214)(235)(245)(123) + (235)(213)(214)(245) = 0, \text{ since } (225) = 0.$$

Similarly writing 4 for 0 throughout, we obtain the identical equation

$$(435)(413)(214)(245) + (425)(413)(435)(214) = 0.$$

Hence the lines 1, 2, 3, 4, 5 all lie in the surface, and it is of the second order, therefore it is the equation to the cones having these lines for sides.

This form of the equation is deduced from Hamilton's condition for six lines forming the sides of a cone of the second degree.

It follows at once from Pascal's Theorem that if $\beta', \beta'', \beta'''$ be the intersections of the planes (1, 2), (4, 5); (2, 3), (5, 6); (3, 4), (6, 1); $\beta'\beta''\beta'''$ lie in a plane, or $S\beta'\beta''\beta''' = 0$.

But
$$\beta' = V \cdot V(1, 2), V(4, 5) = 1S(2, 4, 5) - 2S(1, 4, 5),$$

or $5S(4, 1, 2) - 4S(5, 1, 2);$

$$\beta'' = 2S(3, 5, 6) - 3S(2, 5, 6), \quad \beta''' = 3S(4, 6, 1) - 4S(3, 6, 1);$$

$$\therefore V\beta'\beta''\beta''' = S(3, 5, 6), S(4, 6, 1), V(2, 3) - S(3, 5, 6), S(3, 6, 1), V(2, 4) \\ + S(2, 5, 6), S(3, 6, 1), V(3, 4);$$

$$\therefore S\beta'\beta''\beta''' = S(2, 4, 5), S(3, 5, 6), S(4, 6, 1), S(1, 2, 3) \\ - S(2, 4, 5), S(3, 5, 6), S(3, 6, 1), S(1, 2, 4) \\ + S(4, 1, 2), S(2, 5, 6), S(3, 6, 1), S(5, 3, 4),$$

using the second form for β' to obtain the last product.

Hence, replacing 6 by 0, and writing (245) &c., for $S(2, 4, 5)$, we have (014)(035)(245)(123) + (035)(013)(214)(245) + (025)(013)(435)(214) = 0; the expression required.

COR. 1. If six planes through a point touch a cone of the second order, their normals at this point are six sides of a cone.

This is easily seen from the Cartesian equation to a cone, but it follows from the above condition.

For let $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6$ be the six normals to the planes 1, 2, 3, 4, 5, 6.

Then the common section of 1 and 2 is perpendicular to $\gamma_1\gamma_2$; and therefore common section is parallel to $V\gamma_1\gamma_2$. Similarly common section of 4 and 5 is parallel to $V\gamma_4\gamma_5$.

But by Brianchon's Theorem the planes containing the common sections of (1, 2), (4, 5); (2, 3), (5, 6); (3, 4), (6, 1) meet in a line; therefore their normals through the vertex lie in a plane; therefore $S\beta'\beta''\beta''' = 0$, where $\beta', \beta'', \beta'''$ are these normals.

But $\beta' = V.V\gamma_1\gamma_2.V\gamma_4\gamma_5$, &c., which is the same form of condition as obtained above.

Cor. 2. By comparing the condition of the question with the result of eliminating the constants from the six equations obtained by writing the coordinates of the points in the Cartesian equation

$$Ax^2 + By^2 + Cz^2 + Fyz + Gzx + Hxy = 0,$$

we obtain an expression for a determinant of the 6th order in terms of determinants of the 3rd order.

It would be interesting to see Pascal's Theorem deduced from Hamilton's form of the equation to a cone $S \frac{a}{\rho} S \frac{\rho}{\beta} = 1$, but I am not sufficiently acquainted with the method of quaternions to apply the equation in this way.

3741. (Proposed by Professor TOWNSEND, F.R.S.)—Show that, when the curve of intersection of two quadric surfaces lies on a sphere, both surfaces have the same two systems of cyclic planes; and, conversely, that, when two quadric surfaces have the same two systems of cyclic planes, their curve of intersection lies on a sphere.

Solution by R. F. SCOTT; X. Y. Z.; and others.

Let ϕ be one quadric, then the other may be written $\phi - S = 0$ where S is a sphere. Any pair of circular sections of the first may be written $\phi - \lambda S' = 0$, where S' is another sphere. Clearly $\phi - S - \lambda S' + S = 0$ represents the same pair of planes, but they are circular sections of $\phi - S = 0$. Conversely, if the two quadrics have the same circular section, they may be written $S - uv = 0$, $S' - uv = 0$. Subtracting, we see they intersect in the sphere $S - S' = 0$.

II. Solution by the PROPOSER.

For, as regards the first part, as every arbitrary plane intersects the two quadrics and the sphere in two conics and a circle passing through the four points in which it intersects their common curve, therefore every plane which intersects either quadric in a circle intersects the other in a circle also. And, as regards the second part, as the two quadrics intersect two different planes which are not parallel each in a pair of circles, their curve of intersection passes consequently through, and lies therefore upon the sphere determined by, the eight points of intersection of the two pairs of circles, and therefore, &c.

III. *Solution by J. J. WALKER, M.A.*

1. Let one of the surfaces be $ax^2 + by^2 + cz^2 + d = 0$, then the directions of the cyclic planes will be determined by $\frac{b-a}{c-b}$, suppose. The equation to a second quadric intersecting the former in the sphere

$$x^2 + y^2 + z^2 + 2lx + 2my + 2nz + r = 0$$

will be $(a+\lambda)x^2 + (b+\lambda)y^2 + (c+\lambda)z^2 + 2\lambda lx + 2\lambda my + 2\lambda nz + d + \lambda r = 0$, the directions of the cyclic planes of which will be determined by

$$\frac{b+\lambda-(a+\lambda)}{c+\lambda-(b+\lambda)} = \frac{b-a}{c-b}.$$

Or if one surface be a paraboloid, $ax^2 + by^2 + 2cz = 0$, the directions of the cyclic planes will be determined by $\frac{a-b}{b}$, suppose. The equation to the other quadric will now be of the form

$$(a+\lambda)x^2 + (b+\lambda)y^2 + \lambda z^2 + \dots = 0,$$

and the directions of its cyclic planes be determined by

$$\frac{b+\lambda-(a+\lambda)}{\lambda-(b+\lambda)} = \frac{a-b}{b}.$$

2. Through any point in the curve of intersection of two quadric surfaces S, S' let a plane L be drawn cutting S in a circle, then the same plane must cut S' in a circle, and these two circles having at least three points common must coincide; the common section of S, S' is therefore two circles, which lie on a sphere.

3859. (Proposed by S. WATSON.)—Two equal spheres intersect: show that the chance of the portion common to both exceeding half the volume of either, is $8 \cos^3 80^\circ$.

I. *Solution by C. LEUDESDOFF.*

Suppose one sphere (centre O , radius a) fixed, and cut by the other sphere in a circle whose projection on the plane of the paper (which passes through its centre) is the straight line PMQ . Let $OM = x$; then

Half volume of either sphere $= V_1 = \frac{2}{3}\pi a^3$,

$$\begin{aligned} \text{Volume common to two spheres} &= V = 2\pi \int_0^a (a^2 - x^2) dx \\ &= \frac{2}{3}\pi a^3 (2 + 2 \cos 3\theta), \text{ (if } x \equiv 2a \cos \theta). \end{aligned}$$

Hence if V be not $> V_1$, we must have $\cos 3\theta > -\frac{1}{2}$, and θ may lie between 40° and 80° . But because x may lie between a and $-a$, the extreme values of θ are 60° and 120° . Hence in order that V be not $> V_1$, M must lie within the space bounded by two spheres of centre O and radii $2a \cos 60^\circ$ and $2a \cos 80^\circ$ respectively. Therefore the chance that $V > V_1$ is

$$1 - \frac{\frac{4}{3}\pi \{(2a \cos 60^\circ)^3 - (2a \cos 80^\circ)^3\}}{\frac{4}{3}\pi a^3} = 8 \cos^3 80^\circ.$$

II. *Solution by ARTEMAS MARTIN.*

Let $2x$ = distance between the centres of the spheres when the portion common to both is one-half of either; then the radii of the spheres being taken equal to unity, the probability required is x^3 , and x is to be determined from the cubic equation $x^3 - 3x = -1$. Solving by Trigonometry, we have $x = 2 \sin 10^\circ = 2 \cos 80^\circ$; therefore $p = 8 \cos^3 80^\circ$.

3841. (Proposed by W. SILVERLY.)—A bowl which is a segment of a hollow sphere whose external and internal radii are R and r , and depth $r - c$, rests on a horizontal plane, and has a weight attached to the outer edge of the rim is one n th the weight of the bowl. Show that in its position of equilibrium the inclination of the rim of the bowl to the horizon is

$$\tan^{-1} \frac{4(R^2 - c^2)^{\frac{1}{2}} \{ (2R + c)(R - c)^2 - (2r + c)(r - c)^2 \}}{3n \{ (R^2 - c^2)^2 - (r^2 - c^2)^2 \} + 4c \{ (2R + c)(R - c)^2 - (2r + c)(r - c)^2 \}}.$$

Solution by the PROPOSER; A. B. EVANS, M.A.; and others.

Let O be the centre of the hollow sphere, G the centre of gravity of the bowl, W the weight attached at A , nW the weight of the bowl, and the inclination of the rim of the bowl to the horizon, $\angle AOG = \phi$, and $OG = x$. Then taking moments about O , we have

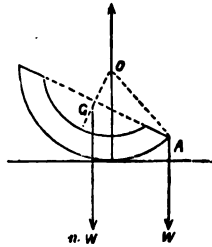
$$\begin{aligned} nx \sin \theta &= R \sin(\phi - \theta) \\ &= R(\sin \phi \cos \theta - \cos \phi \sin \theta) \dots\dots (1). \end{aligned}$$

But $\sin \phi = \frac{(R^2 - c^2)^{\frac{1}{2}}}{R}$, and $\cos \phi = \frac{c}{R}$; hence

$$\text{from (1) we have } \tan \theta = \frac{(R^2 - c^2)^{\frac{1}{2}}}{nx + c} \dots\dots\dots (2).$$

$$\begin{aligned} \text{But } x &= \frac{\int_c^R (R^2 - x^2) x dx - \int_c^r (r^2 - x_1^2) x_1 dx_1}{\int_c^R (R^2 - x^2) dx - \int_c^r (r^2 - x_1^2) dx_1} \\ &= \frac{3}{4} \left(\frac{(R^2 - c^2)^2 - (r^2 - c^2)^2}{(2R + c)(R - c)^2 - (2r + c)(r - c)^2} \right), \end{aligned}$$

which, when substituted in (2), gives the result in the question.



3872. (Proposed by W. C. OTTER, F.R.A.S.)—Given a circle and a point not in the plane of the circle; find the centre of the sphere which passes through the given point and through the circumference of the given circle.

Solution by J. MERRIFIELD, LL.D.; A. B. EVANS, M.A.; and others.

Let $abcd$ be the given circle, and P the given point. Find the centre of the given circle and draw AC from the centre, perpendicular to its plane, then AC shall pass through the centre of the required sphere.

Draw a plane which shall contain the straight line AC , and let it revolve about AC until it pass through P ; this plane must necessarily cut the given circle $abcd$ in two points as at b and d . We have now three points in a great circle of the required sphere. Join Pb by a straight line, bisect it at right angles by the line MN ; then the point where MN meets AC is the centre of the required sphere.

3794. (Proposed by R. TUCKER, M.A.)— AB, AC are two given indefinite straight lines touching a given circle in E and F ; a variable line BC is drawn touching the circle, and BF, CE are drawn intersecting in P ; show that the locus of P is an ellipse also touching the given lines AB, AC in E and F .

I. Solution by M. COLLINS, B.A.

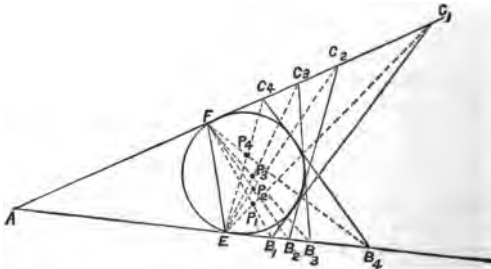
Since the variable tangent BC subtends a constant angle at the centre of a circle (or at a focus of a conic) touching AB, AC at E and F , therefore, by taking four positions of BC , we have the anharmonic ratio $E(C_1C_2C_3C_4) = F(B_1B_2B_3B_4)$; and therefore the conic passing through $EEF, P_1P_2P_3P_4$ must also pass through P_4 , the point of intersection of EC_4 and FB_4 ; moreover, when B is infinitely near A , P is infinitely near F and almost upon AF ; therefore the locus of P is a conic touching AE and AF at E and F , and this is true even when the original circle is any conic touching AE and AF at E and F .

[In this Question any conic may replace the circle, and the two points E, F may be any whatever. For if two fixed tangents be drawn to a conic and any other tangent meet them in a, a' ; then a, a' are corresponding points of two homographic ranges, which will therefore subtend homographic pencils at any two points whatever O, O' ; and the intersection of pairs of corresponding rays is a conic through O, O' , and touching at O, O' the rays corresponding to $O'O, OO'$ respectively. The number of loci and envelopes which depend at once on this simple principle is uncountable.]

II. Solution by C. LEUDESDOFF; R. W. GENESE, B.A.; and others.

Take four positions of BC as in the figure; then since the lines AEB, AFC may be considered as a conic,

$$\begin{aligned} E(C_1C_2C_3C_4) \\ &= F(B_1B_2B_3B_4); \\ \text{therefore} \\ E(P_1P_2P_3P_4) \\ &= F(P_1'P_2'P_3'P_4'), \end{aligned}$$



and therefore $P_1P_2P_3P_4$ lie on a conic passing through E and F. And the tangent to this conic at E is the line of the pencil E($P_1 \dots$) corresponding to FE considered as a line of the pencil F($P_1 \dots$), which is seen by the construction to be AB. Hence the locus touches AB in E, and can in the same way be shown to touch AC in F.

III. Solution by J. DALE; A. ESCOTT, M.A.; and others.

Instead of a circle, let $ax^2 + 2a'yz = 0$ be the equation of any conic touching AE, AF in E and F, and let $lx + my + nz = 0$ be the equation of any variable tangent BC, then BE, CF intersect in the point

$$\frac{x}{-mn} = \frac{y}{nl} = \frac{z}{lm};$$

and because BC is a tangent, $2amn + a'l^2 = 0$; therefore the locus of P is the conic $2ax^2 + a'yz = 0$. When, as in the present case, the first conic is a circle, the locus of P is the ellipse $x^2 = 4yz$.

2671. (Proposed by H. S. HALL.)—If p_n denotes the ratio of the product of the first n odd numbers to the product of the first n even numbers, prove that

$$\begin{aligned} (1) \dots p_{2m} - p_1 p_{2m-1} + p_2 p_{2m-2} - \dots + (-1)^{m-1} p_{m-1} p_{m+1} \\ = \frac{1}{2} p_m \{ 1 - (-1)^m p_m \} \\ (2) \dots \frac{p_{2m}}{4m-1} + \frac{p_1}{1} \cdot \frac{p_{2m-1}}{4m-3} - \frac{p_2}{3} \cdot \frac{p_{2m-2}}{4m-5} + \dots + (-1)^m \frac{p_{m-1}}{2m-3} \cdot \frac{p_{m-1}}{2m+1} \\ - \frac{\frac{1}{2} p_m}{2m-1} \left\{ 1 + (-1)^m \frac{p_m}{2m-1} \right\}. \end{aligned}$$

Solution by the PROPOSER.

1. The coefficient of x^n in the expansion of $(1-x)^{-\frac{1}{2}}$ is

$$\frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2 \cdot 4 \cdot 6 \dots 2n}, \text{ or } p_n.$$

Hence $(1-x)^{-\frac{1}{2}} = 1 + p_1 x + p_2 x^2 + p_3 x^3 + \dots + p_{2m} x^{2m} + \dots$,

and $(1+x)^{-\frac{1}{2}} = 1 - p_1 x + p_2 x^2 - p_3 x^3 + \dots + p_{2m} x^{2m} - \dots$;

$$(1-x^2)^{-\frac{1}{2}} = \&c. + x^{2m} \{ p_{2m} - p_1 p_{2m-1} + p_2 p_{2m-2} - \dots - p_{2m-1} p_1 + p_{2m} \};$$

the coefficient of x^{2m} in the expansion of $(1-x^2)^{-\frac{1}{2}}$ is p_m ,

therefore $p_{2m} - p_1 p_{2m-1} + p_2 p_{2m-2} - \dots - p_{2m-1} p_1 + p_{2m} = p_m$;

and there are $2m+1$ terms in the above series.

Hence $2 \{ p_{2m} - p_1 p_{2m-1} + p_2 p_{2m-2} - \dots \text{to } m \text{ terms} \} + (-1)^m p_m p_m = p_m$ therefore

$$p_{2m} - p_1 p_{2m-1} + p_2 p_{2m-2} - \dots + (-1)^{m-1} p_{m-1} p_{m+1} = \frac{1}{2} p_m \{ 1 - (-1)^m p_m \}$$

It is easily shown (*Vide* TODHUNTER'S *Algebra*, Art. 521) that

$$(1-x)^{\frac{1}{2}} = 1 - p_1x - \frac{1}{2}p_2x^2 - \frac{1}{2}p_3x^3 - \dots - \frac{1}{4m-1}p_{2m}x^{2m} - \dots,$$

$$(1+x)^{\frac{1}{2}} = 1 + p_1x - \frac{1}{2}p_2x^2 + \frac{1}{2}p_3x^3 - \dots - \frac{1}{4m-1}p_{2m}x^{2m} + \dots,$$

$$(1-x^2)^{\frac{1}{2}} = \&c. - x^{2m} \left\{ \frac{p_{2m}}{4m-1} + \frac{p_1 p_{2m-1}}{1 \cdot 4m-3} - \dots \text{to } 2m+1 \text{ terms} \right\}.$$

Then, proceeding as in the former case, we obtain the required theorem.

3820 (Proposed by N'IMPORTE).—From a given point D in the produced diameter BA of a given circle ACP, to draw a line DCH, cutting the circumference in C and H, so that, if BC, BH be joined, the area of the triangle BCH may be given or a maximum.

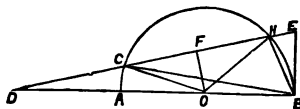
Solution by H. MURPHY; R. W. GENESE, B.A.; and others.

From the centre O draw OF perpendicular to DH, and BE parallel to OF; then $\triangle BCH : \triangle OCH$

$$= BE : OF = DB : DO.$$

Hence if BCH be given, the area of the isosceles triangle OCH becomes known, its two sides OC, OH, and its angles COH, OCH; thus C and H are the intersections of the given circle with segments of circles on OD containing determined angles.

That the triangle BCH may be a maximum, COH must be a maximum, and its sides OC and OH being given, therefore COH is a right angle, hence DCO is a right angle and a half, and therefore C becomes known.



II. *Solution by C. LEUDESORF.*

Let $OA = a$, $OD = d$, $\angle ODC = \theta$; then $CH = 2(a^2 - d^2 \sin^2 \theta)^{\frac{1}{2}}$; therefore $\triangle BCH = \frac{1}{2}CH \cdot PA \cdot \sin \theta = (d-a)(a^2 - d^2 \sin^2 \theta)^{\frac{1}{2}} \sin \theta$. Hence $\triangle BCH$ will be a maximum or minimum when $u = \sin^2 \theta (a^2 - d^2 \sin^2 \theta)$ is maximum or minimum. Hence we have

$$\frac{du}{d\theta} = \sin 2\theta (a^2 - 2d^2 \sin^2 \theta) = 0 \text{ when } \theta = 0 \text{ or } \frac{1}{2}\pi, \text{ or } \sin \theta = \frac{a}{d\sqrt{2}}.$$

Now $\theta = 0$ makes Δ vanish, and when $\theta = \frac{1}{2}\pi$, then $\frac{d^2u}{d\theta^2} = 4d^2 - 2a^2$,

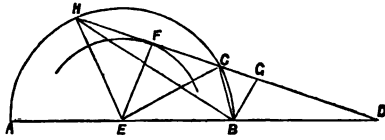
and because $d > a$, $\frac{d^2u}{d\theta^2}$ is positive, hence $\theta = \frac{1}{2}\pi$ gives a minimum.

When $\sin \theta = \frac{a}{d\sqrt{2}} \frac{d^2 u}{d\theta^2} = -\frac{2\sqrt{2}}{d} a^2 (2h^2 - a)^{\frac{1}{2}} = \text{a negative quantity,}$
and therefore this value of θ gives a maximum.

In this case $\Delta BCH = (d-a) \frac{a}{d\sqrt{2}} (a^2 - \frac{1}{2}a^2)^{\frac{1}{2}} = \frac{1}{2} \left(1 - \frac{a}{d}\right) a^2.$

III. Solution by N'IMPORTE.

Find the centro E ; join EC, EH ; and draw the perpendiculars EF, BG on CH . Since the triangles ECH, BCH have the same base CH , they are as their altitudes $EF : BG$ or as $ED : DB$, a given ratio. When the triangle BCH is given the triangle ECF is given; and EC being given and the angle EFC a right angle, EF is given, and a circle from centre E with radius EF is given, and therefore the tangent DFH .



When the triangle ECH is a maximum, and EC, EH given lines, it is well known that $\angle CEH$ is a right angle, and therefore CH a given line; hence in this case it is only to draw DH so that the intercept CH be equal to a given line.

3774. (Proposed by Professor TOWNSEND, F.R.S.)—Show that the two sheets of the biaxial wave surface, apsidal of the ellipsoid whose semi-axes are a, b, c , are reciprocal polars to each other with respect to the coaxial ellipsoid whose corresponding semi-axes are $(bc)^{\frac{1}{2}}, (ca)^{\frac{1}{2}}, (ab)^{\frac{1}{2}}$.

Solution by the PROPOSER.

Denoting by O the common centre of the ellipsoid abc , and of its apsidal, the wave surface in question; by r and r_1 any pair of corresponding central radii of the two surfaces; by p and p_1 the central perpendiculars on the corresponding pair of tangent planes; by q and q_1 the connectors of the feet of p and r and of p_1 and r_1 ; by $\alpha\beta\gamma$ and $\alpha_1\beta_1\gamma_1$ the direction angles of p and p_1 ; by u, v, w the coordinates of any point on the tangent plane to the apsidal; and by x, y, z those of the pole of that plane with respect to the ellipsoid $(bc)^{\frac{1}{2}}, (ca)^{\frac{1}{2}}, (ab)^{\frac{1}{2}}$. Then, since pqr for the ellipsoid abc , turned round O through a right angle in its own plane, coincides in its new position with the triangle $p_1q_1r_1$ for its apsidal, the equation of the tangent plane to the latter is consequently

$$u \cos \alpha_1 + v \cos \beta_1 + w \cos \gamma_1 = p \quad \dots \dots \dots (1),$$

which, identified with that of the polar plane of xyz with respect to the ellipsoid $(bc)^{\frac{1}{2}}, (ca)^{\frac{1}{2}}, (ab)^{\frac{1}{2}}$, viz., $\frac{ux}{bc} + \frac{vy}{ca} + \frac{wz}{ab} = 1 \quad \dots \dots \dots (2),$

gives accordingly, for the determination of x, y, z , the three relations

$$\begin{aligned}\frac{px}{bc} &= \cos \alpha_1 = \frac{a^2 - p^2}{pq} \cos \alpha, & \frac{py}{ca} &= \cos \beta_1 = \frac{b^2 - p^2}{pq} \cos \beta, \\ \frac{pz}{ab} &= \cos \gamma_1 = \frac{c^2 - p^2}{pq} \cos \gamma \dots\dots\dots (3);\end{aligned}$$

and it remains only to be shown that their values, as given by these relations, satisfy the equation (Salmon's *Geometry of Three Dimensions*, 2nd edition, Art. 458) of the apsidal of abc , viz.,

$$[x^2 + y^2 + z^2] [a^2x^2 + b^2y^2 + c^2z^2] - [a^2x^2(b^2 + c^2) + b^2y^2(c^2 + a^2) + c^2z^2(a^2 + b^2)] + [a^2b^2c^2] = 0 \dots\dots (4),$$

which it is easily seen they do; for substituting in the latter equation (4) for a^2x^2, b^2y^2, c^2z^2 , and $x^2 + y^2 + z^2$ their values as given by the preceding equations (3), we get, after some obvious reductions, the relation

$$(b^2 - p^2)(c^2 - p^2) \cos^2 \alpha_1 + (c^2 - p^2)(a^2 - p^2) \cos^2 \beta_1 + (a^2 - p^2)(b^2 - p^2) \cos^2 \gamma_1 = 0$$

or, since from the same equations (3),

$$\cos \alpha_1 : \cos \beta_1 : \cos \gamma_1 = (a^2 - p^2) \cos \alpha : (b^2 - p^2) \cos \beta : (c^2 - p^2) \cos \gamma \dots\dots (6),$$

the equivalent relation

$$[(a^2 - p^2)(b^2 - p^2)(c^2 - p^2)] [(a^2 - p^2) \cos^2 \alpha + (b^2 - p^2) \cos^2 \beta + (c^2 - p^2) \cos^2 \gamma] = 0 \dots\dots (7),$$

which, by virtue of the known relation $p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma$, is evidently true from the vanishing of its last factor.

N.B.—The relation of pole and polar with respect to any quadric surface remaining unchanged by homographic transformation, the above property is consequently true, not only for the apsidal of the ellipsoid abc with respect to the connected ellipsoid $(bc)^{\frac{1}{2}}, (ca)^{\frac{1}{2}}, (ab)^{\frac{1}{2}}$, but more generally for every surface derived from the former by homographic transformation, with respect to the quadric derived from the latter by the same transformation.

3580. (Proposed by T. MITCHELSON, B.A.)—If the magnifying power of the common Astronomical Telescope, whose theoretic magnifying power is μ , becomes μ_1, μ_2 , &c., for ν persons whose distances of distinct vision are δ_1, δ_2 , &c., respectively, and if ϕ = the focal length of the eye-glass, show that

$$\phi = \frac{\delta_1(\mu_1 - \mu) + \delta_2(\mu_2 - \mu) + \&c. + \delta_\nu(\mu_\nu - \mu)}{\mu\nu}.$$

Solution by Professor TOWNSEND, F.R.S.

Since, by the conditions of the question,

$$\mu_1 = \mu \left(1 + \frac{\phi}{\delta_1}\right), \quad \mu_2 = \mu \left(1 + \frac{\phi}{\delta_2}\right), \quad \mu_\nu = \mu \left(1 + \frac{\phi}{\delta_\nu}\right), \quad \&c.,$$

therefore at once, by addition,

$$(\mu_1 - \mu) \delta_1 + (\mu_2 - \mu) \delta_2 + (\mu_\nu - \mu) \delta_\nu + \&c. = \mu\nu\phi;$$

and therefore, &c.

3906. (Proposed by Rev. Dr. BOOTH, F.R.S.)—The centres of a series of circles range along a parabola, their focal distances are f , and their radii are $(f^2 - a^2)^{\frac{1}{2}}$, $4a$ being the parameter; find the curve which envelopes the circles.

Solution by PROFESSOR WOLSTENHOLME; A. M. NASH; M. COLLINS, M.A.; and others.

The coordinates of the centre of one of the circles being $am^2, 2am$, its equation will be
 $(x - am^2)^2 + (y - 2am)^2 = a^2(1 + m^2)^2 - a^2$,
 or $m^2(2a^2 - 2ax) - 4amy + x^2 + y^2 = 0$;
 whence the envelope is the well known circular cubic $2ay^2 = (x^2 + y^2)(a - x)$,

$$\text{or} \quad y^2 = x^2 \frac{a - x}{a + x};$$

or, in polar coordinates $r \cos \theta = a \cos 2\theta$;
 which is a curve somewhat like EAFAD in the figure, where BAC is the parabola, F the focus, and HKL the directrix.

Also for the two points of contact of the circle with its envelope $m(a - x) = y$, or they both lie on the focal perpendicular upon the tangent to the parabola at the centre of the circle. The tangents to the circle at these points meet in the point (X, Y) , where (X, Y) are determined by the condition that $(x - am^2)(X - am^2) + (y - 2am)(Y - 2am) = a^2(2m^2 + m^4)$ shall coincide with

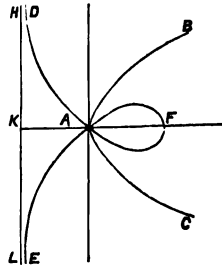
$$y + mx = ma;$$

$$\text{therefore} \quad Y - 2am = \frac{X - am^2}{m} = m(X - am^2) + 2(Y - 2am) + a(2m + m^3)$$

$$= mX + 2Y - 2am,$$

$$\text{or} \quad Y + mX = 0, \quad Y = am + \frac{X}{m};$$

which are the equations giving the foot of the perpendicular from the vertex upon the tangent at the centre of the circle. The locus of the pole with respect to each circle of its chord of contact with the envelope is therefore the cissoid which is the pedal of the parabola with respect to the vertex.



3604. (Proposed by the EDITOR.)—Find x and y from the equations
 $x + 2y - xy^2 + a(1 - 2xy - y^2) = y + 2x - x^2y + b(1 - 2xy - x^2) = 0$.

Solution by the PROPOSER.

The equations may be put into the form

$$\frac{2y}{1 - y^2} + \frac{x + a}{1 - ax} = 0, \quad \frac{2x}{1 - x^2} + \frac{y + b}{1 - by} = 0;$$

and if we put $a = \tan \alpha$, $b = \tan \beta$, $x = \tan \theta$, $y = \tan \phi$, they become

$$\tan 2\phi + \tan (\alpha + \theta) = 0, \quad \tan 2\theta + \tan (\beta + \phi) = 0;$$

the solution of which is given by $2\phi + \alpha + \theta = n\pi = 2\theta + \beta + \phi$, where n is any whole number. We thus obtain

$$x = \tan \theta = \tan \frac{1}{2}(n\pi + \alpha - 2\beta), \quad y = \tan \phi = \tan \frac{1}{2}(n\pi + \beta - 2\alpha);$$

where the only values n can have, so as to give *different* values for x and y , are 0, 1, 2.

3885. (Proposed by Professor CLIFFORD.)—If A be the single focus of a semi-cubical parabola, there exists a straight line BC , such that if two tangents at right angles cut it in B, C , the angle BAC is also a right angle.

I. Solution by A. M. NASH.

The focus of the semi-cubical parabola is also the focus of the parabola of which the given curve is the evolute; and since normals to a curve are tangents to its evolute, the theorem is equivalent to the following one:—If A be the focus of a parabola, there exists a straight line such that if two normals at right angles cut it in B, C , the angle BAC is also a right angle.

Reciprocating this theorem with respect to the focus, and remembering that tangents at right angles intersect on the directrix, and that the normals are parallel to the tangents, we get the following theorem:

If O be a fixed point on a circle, of which C is the centre; AE, BD any pair of parallel tangents, and if AO meet BD in D , and BO meet AE in E , then DE will always subtend a right angle at a fixed point (other than O).

Take OC and the tangent at O as axes, and let $OC = a$, and let (x', y') be the coordinates of A ; then the coordinates of B will be $(2a - x', -y')$; and the equations of AOB, AE, BD, AO, BO will be respectively

$$x^2 + y^2 = 2ax, \quad x(x' - a) + yy' = ax' \dots (1, 2),$$

$$x(x' - a) + yy' = a(x' - 2a), \quad y = \frac{y'}{x'}x, \quad y = \frac{y'}{x' - 2a}x \dots (3, 4, 5).$$

From (3), (4), and (2), (5) we get, for the coordinates of D, E ,

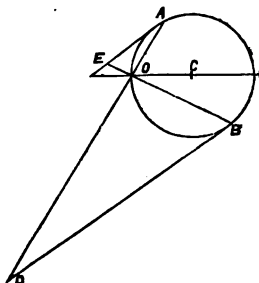
$$x_1 = x' - 2a, \quad y_1 = \frac{y'}{x'}(x' - 2a); \quad x_2 = -x', \quad y_2 = -\frac{x'y'}{x' - 2a}.$$

Hence the equation of the circle on DE as diameter is

$$\left\{ x - (x' - 2a) \right\} (x + x') + \left\{ y - \frac{y'}{x'}(x' - 2a) \right\} \left(y + \frac{x'y'}{x' - 2a} \right) = 0,$$

$$\text{which reduces to} \quad x^2 + y^2 + 2ax - \frac{4a(a - x')}{y'}y = 0.$$

Now this circle evidently passes through the fixed point $x = -2a, y = 0$.



And on reciprocating back again we find that the required line BC is perpendicular to the axis, which it cuts in a point as far from the focus on one side as the vertex is on the other; and this point is the vertex of the semi-cubical parabola.

II. Solution by C. LEUDES DORF.

Reciprocating the semicubical parabola $27ay^2 = 4(x+a)^3$ with respect to the circle (centre A) $x^2 + y^2 = a^2$, we obtain the cissoid

$$y^2 = \frac{x^3}{a-x} \dots\dots\dots (1).$$

The problem then becomes the following: To show that there exists a point O such that any two points P(x_1, y_1) and Q(x_2, y_2) lying on (1), which subtend a right angle at the origin, subtend also a right angle at O.

Because $\angle PAQ = 90^\circ$, therefore $x_1x_2 + y_1y_2 = 0 \dots\dots\dots (2),$

hence, by (1),
$$x_1x_2 = \frac{x_1^3x_2^3}{(a-x_1)(a-x_2)};$$

and therefore
$$x_1 + x_2 = a \dots\dots\dots (3).$$

And by (2) and (3) $(2a-x_1)(2a-x_2) + y_1y_2 = 0,$

which shows that P, Q subtend also a right angle at O ($2a, 0$).

PROOF OF PASCAL'S THEOREM. By F. D. THOMSON, M.A.

The following proof of Pascal's Theorem as to a hexagon in a conic is new to me, and requires no knowledge of any properties of the curve.

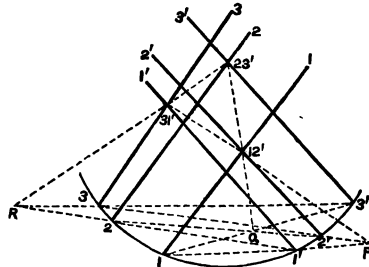
Consider the surface generated by a straight line which always meets three fixed non-intersecting lines.

Let 1, 2, 3 be the three fixed lines, and 1', 2', 3' positions of the moving line.

Then the common section of the planes containing 3', 1; 2, 2' passes through the points (3', 2), (1, 2'); the common section of the planes containing 1', 2; 3, 3' passes through the points (1', 3), (2, 3'); the common section of the planes containing 2', 3; 1, 1' passes through the points (2', 1), (3, 1'). Or, the three common sections form a triangle, and therefore lie in a plane.

Now let the surface be cut by any arbitrary plane.

The cutting plane will meet the planes 3', 1; 2, 2'; &c., in a hexagon 1, 1', 2, 2', 3, 3', and the three common sections, being co-planar, will meet the cutting plane in a straight line PQR, P, Q, R being also the intersections of opposite sides of the hexagon.



This is Pascal's Theorem, which is thus proved to be true of any plane section of the ruled surface.

Cor.—By joining all the points of the plane figure to an arbitrary point in space, it follows at once that the Theorem is true for any plane section of an oblique cone which has the curve for its base.

Hence the whole theory of conic sections may be deduced; for Pascal's Theorem, like the general equation, is a means of expressing the relation between any six points on the curve.

This method of proof seems to show very clearly, not only that the theorem is true, but *why* it is so.

3883. (Proposed by M. COLLINS, B.A.)—Prove that the harmonic mean between the segments of any chord of a conic section passing through its focus is constant and equal to the semi-parameter.

Solution by the Honourable ALGERNON BOURKE.

Let $SP_1 (=r_1)$ and $SP_2 (=r_2)$ be the segments of the focal chords P_1SP_2 , and p the semi-parameter; then the harmonic mean between r_1 and r_2 is $\frac{2r_1r_2}{r_1+r_2}$. Now $r_1 = \frac{p}{1+\epsilon \cos \alpha}$, and $r_2 = \frac{p}{1-\epsilon \cos \alpha}$; hence, by substituting these values of r_1, r_2 we obtain p for the harmonic mean.

3900. (Proposed by W. H. H. HUDSON, M.A.)—Six equal and similar unstretched light elastic strings are fastened together so as to form a square, and the two lines bisecting opposite sides. Four equal forces being applied at the angular points of the square along the diagonals outwards, obtain equations to determine the shape and size of the network.

Solution by Professor TOWNSEND, F.R.S.

The network, in its form of equilibrium, being evidently an octagonal star, symmetrical on both sides of the four connectors of its opposite vertices, and concentric with their common intersection; denoting by α and β the common values of its four equal internal semi-salient and external semi-reentrant angles respectively, by x and y those of the semi-extensions of the four sides and of the two bisectors of the original square, by l the common original semi-length of the six lines, and by k the ratio of the common value F of the four equal straining forces to the common modulus of elasticity E of the six lines; we have, on the ordinarily received principles, for the determination of α and β, x and y , the four following equa-

tions $x = \frac{1}{2}kl \sec \alpha$, $y = 2x \cos \beta = kl \sec \alpha \cos \beta$,

$$(l+x) \sin \alpha = (l+y) \sin \beta, \quad \beta = \alpha + \frac{1}{2}\pi,$$

which give immediately x and y in terms of α , and for the latter, by elimination, the equation $\sin \alpha + k \tan \alpha = \sqrt{\frac{1}{2}}(1 + k\sqrt{\frac{1}{2}})$,

which shows that the extreme values of $\tan \alpha$, for $k=0$ and $=\infty$, are 1 and $\frac{1}{2}$ respectively, but which cannot be solved generally in finite terms for intermediate values of k .

3860. (Proposed by W. H. H. HUDSON, M.A.)—If

$$\frac{\cos(\beta + \alpha) + \cos(\alpha + \gamma)}{\cos(\beta - \alpha) + \cos(\alpha - \gamma)} = \frac{\cos(\beta + \gamma) + \cos(\alpha + \gamma)}{\cos(\gamma - \beta) + \cos(\alpha - \gamma)},$$

prove that
$$\frac{\sin \beta}{\cos \gamma - \cos \alpha} + \frac{\cos \beta}{\sin \alpha - \sin \gamma} = \frac{1}{\sin(\alpha - \gamma)}.$$

Solution by J. MERRIFIELD, LL.D.; T. MITCHESON, B.A.; and others.

Adding and subtracting the numerators and denominators, we have

$$\frac{\cos \alpha (\cos \beta + \cos \gamma)}{\sin \alpha (\sin \beta + \sin \gamma)} = \frac{\cos \gamma (\cos \beta + \cos \alpha)}{\sin \gamma (\sin \beta + \sin \alpha)};$$

therefore
$$\frac{\cos \frac{1}{2}(\beta + \gamma) \sin \frac{1}{2}(\beta + \alpha)}{\sin \frac{1}{2}(\beta + \gamma) \cos \frac{1}{2}(\beta + \alpha)} = \frac{\sin \alpha \cos \gamma}{\cos \alpha \sin \gamma}.$$

Again adding and subtracting numerators and denominators, we have

$$\frac{\sin \{ \beta + \frac{1}{2}(\alpha + \gamma) \}}{\sin \frac{1}{2}(\alpha - \gamma)} = \frac{\sin(\alpha + \gamma)}{\sin(\alpha - \gamma)};$$

whence we obtain

$$\begin{aligned} \frac{1}{\sin(\alpha - \gamma)} &= \frac{\sin \{ \beta + \frac{1}{2}(\alpha + \gamma) \}}{\sin \frac{1}{2}(\alpha - \gamma)} \\ &= \frac{\sin \beta}{2 \sin \frac{1}{2}(\alpha + \gamma) \sin \frac{1}{2}(\alpha - \gamma)} + \frac{\cos \beta}{2 \cos \frac{1}{2}(\alpha + \gamma) \sin \frac{1}{2}(\alpha - \gamma)} \\ &= \frac{\sin \beta}{\cos \gamma - \cos \alpha} + \frac{\cos \beta}{\sin \alpha - \sin \gamma}. \end{aligned}$$

3839. (Proposed by A. B. EVANS, M.A.)—Prove geometrically the following relations of a plane triangle,

$$p_1 + p_2 + p_3 = R + r, \quad -p_1 + p_2 + p_3 = -R + r_1 \quad \dots\dots\dots (1, 2),$$

$$p_1 - p_2 + p_3 = -R + r_2, \quad p_1 + p_2 - p_3 = -R + r_3 \quad \dots\dots\dots (3, 4);$$

where R denotes the radius of the circumscribed circle; r the radius of the inscribed circle; r_1, r_2, r_3 the radii of the escribed circles; p_1, p_2, p_3 the perpendiculars from the centre of the circumscribed circle on the sides.

Solution by HUGH MURPHY.

Let O be the centre of the circumscribed circle; O_1 that of inscribed circle, and O_2 the centre of one of the escribed circles; and let the perpendiculars OP_1, OP_2, OP_3 be denoted by p_1, p_2, p_3 . Now from the inscribable quadrilaterals, we have, by Ptolemy's theorem,

$$\frac{1}{2}aR = \frac{1}{2}p_2c + \frac{1}{2}p_3b, \quad \frac{1}{2}bR = \frac{1}{2}p_1c + \frac{1}{2}p_3a,$$

$$\frac{1}{2}cR = \frac{1}{2}p_1b + \frac{1}{2}p_2a;$$

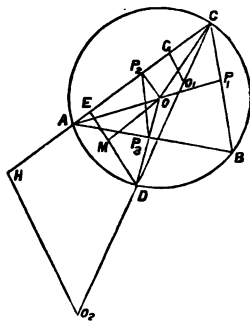
$$\text{also} \quad rs = \frac{1}{2}p_1a + \frac{1}{2}p_2b + \frac{1}{2}p_3c;$$

therefore by addition,

$$(R+r)s = p_1s + p_2s + p_3s.$$

$$\text{Hence} \quad R+r = p_1 + p_2 + p_3 \dots\dots\dots (1).$$

Through O draw OM parallel to AC ; then $ME = p_2$; and from the equality of the triangles DMO and OP_1C , we have $DM = OP_1 = p_1$; but $O_1G + O_2H = 2DE$, that is $r_1 + r = 2p_1 + 2p_2$, from which, if we take (1), we get $-R+r = -p_1 + p_2 + p_3$, and similarly for the others.



3871. (Proposed by the EDITOR.)—In how many different ways may a bookshelf 4 feet 2 inches long be *exactly* filled with volumes $2\frac{1}{8}$, $2\frac{3}{8}$, $2\frac{5}{8}$ inches thick respectively, using at least *one* volume of each sort?

Solution by A. MARTIN; A. B. EVANS, M.A.; J. H. SLADE; Q. W. KING; T. MITCHESON, B.A.; and others.

Let x, y, z represent the number of books $2\frac{1}{8}$, $2\frac{3}{8}$, $2\frac{5}{8}$ inches thick respectively, required to fill the shelf; then we have

$$2\frac{1}{8}x + 2\frac{3}{8}y + 2\frac{5}{8}z = 50, \text{ or } 17x + 19y + 21z = 400 \dots\dots\dots (1).$$

It is evident that the greatest integral value of z in (1) is $z = 17$.

$$\text{Also} \quad x = 24 - y - z - \frac{2(4 + y + 2z)}{17},$$

and putting $\frac{4 + y + 2z}{17} = \text{a whole number} = v$, we have

$$y = 17v - 2z - 4, \quad x = z - 19v - 28 \dots\dots\dots (2).$$

Give z , for example, the values 1, 5, 7; then (2) becomes

$$\begin{array}{l|l|l} y = 17v - 6 & y = 17v - 14 & y = 17v - 18 \\ x = 29 - 18v & x = 33 - 19v & x = 35 - 19v \end{array}$$

in which pairs of equations, the only admissible values of v are 1, 1, 0 respectively. Proceeding thus, we find the following ten solutions, which are all the problem admits of:—

values of $x =$	1	2	3	4	5	6	11	12	13	14,
„ $y =$	11	9	7	5	3	1	8	6	4	2,
„ $z =$	10	11	12	13	14	15	1	2	3	4.

3808. (Proposed by C. TAYLOR, M.A.)—In the right circular cone, if C, C' be the centres of a pair of focal spheres, show that the sphere on CC' contains the auxiliary circle of the section.

Solution by R. W. GENESE, B.A.

Let S, S' be the foci, O the centre of the conic section. A straight line through O at right angles to the plane of section will cut CC' in its middle point, M say. Now let Y be a point on auxiliary circle, then $MY^2 = MO^2 + OY^2 = \text{constant}$. Again if AA' be the major axis, V the vertex of the cone, C and C' are the centres of the inscribed and escribed circles of VAA' ; therefore CAC' is a right angle and $MA = MC$. Thus $MY = MA$, and the locus of Y is a section of the sphere on CC' as diameter.

3890. (Proposed by Professor WOLSTENHOLME.)—Forces P, Q, R, P', Q', R' act along the edges BC, CA, AB, DA, DB, DC of a tetrahedron: prove that they will be equivalent to a couple only if

$$\frac{P'}{AD} = \frac{R}{AB} - \frac{Q}{CA}, \quad \frac{Q'}{BD} = \frac{P}{BC} - \frac{R}{AB}, \quad \text{and} \quad \frac{R'}{CD} = \frac{Q}{CA} - \frac{P}{BC}.$$

I. Solution by F. D. THOMSON, M.A.

The six sides of the tetrahedron may be taken to be the six vectors $a, \beta, \gamma, \gamma - \beta, a - \gamma, \beta - a$, and the six forces P, Q, R, P', Q', R' may be represented by the six vectors $xa, y\beta, z\gamma, x'(\gamma - \beta), y'(a - \gamma), z'(\beta - a)$. Now applying at the point D forces equal and opposite to these six forces, we see that they reduce to the six forces acting at D , and a couple.

Hence the six forces applied at D must be in equilibrium.

Hence since forces are compounded like vectors,

$$xa + y\beta + z\gamma + x'(\gamma - \beta) + y'(a - \gamma) + z'(\beta - a) = 0,$$

or

$$(x + y' - z')a + (y - x' + z')\beta + (z + x' - y')\gamma = 0;$$

therefore since a, β, γ are not co-planar, each term separately vanishes;

therefore $x + y' - z' = 0, \quad y - x' + z' = 0, \quad z + x' - y' = 0,$

which is equivalent to the conditions of the question.

The axis of the couple formed by $-P'$ and $+P'$ acting at D and along BC is evidently $x'\sqrt{\beta}(\gamma-\beta)$ or $x'\sqrt{\beta}\gamma$; and therefore the axis of the resultant couple is $x'\sqrt{\beta}\gamma + y'\sqrt{\gamma\alpha} + z'\sqrt{\alpha\beta}$.

II. *Solution by Professor TOWNSEND, F.R.S.*

For, transferring the three forces P, Q, R to the vertex D of the tetrahedron, thus giving rise to three couples in the three forces BDC, CDA, ADB, which compound a single couple, and resolving the three transferred forces along the three pairs of coplanar edges DB and DC, DC and DA, DA and DB respectively, we have along the three edges DA, DB, DC, the three groups of forces

$$\left(P' + Q \frac{DA}{CA} - R \frac{DA}{AB}\right), \left(Q' + R \frac{DB}{AB} - P \frac{DB}{BC}\right), \left(R' + P \frac{DC}{BC} - Q \frac{DC}{CA}\right),$$

the sum of each of which, for a resultant couple unaccompanied by a resultant single force, must separately be equal to zero, and therefore, &c.

3788. (Proposed by J. J. WALKER, M.A.)—At a point where the normal to the central conic $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axis, a line is drawn parallel to the tangent. Show that the reciprocal polar to the envelope of this line, with respect to the circle described on the line joining the foci as diameter, is $a^2x^2 + b^2y^2 = x^4$.

Solution by JAMES DALE.

The equation to the line through the foot of the normal at h, k , and parallel to the tangent at h, k , is $b^2hx + a^2ky = b^2c^2h^2$. The pole of this line with respect to the circle $x^2 + y^2 = a^2c^2$ is

$$\frac{x}{b^2h} = \frac{y}{a^2k} = \frac{a^2}{b^2h^2}, \quad \text{therefore} \quad \frac{h}{a} = \frac{a}{x}, \quad \frac{k}{b} = \frac{by}{x^2};$$

therefore $\frac{a^2}{x^2} + \frac{b^2y^2}{x^4} = \frac{h^2}{a^2} + \frac{k^2}{b^2} = 1$, or $a^2x^2 + b^2y^2 = x^4$.

3924. (Proposed by B. WILLIAMSON, M.A.)—Prove that the eccentricity of any section of a right cone is represented by $\frac{\cos \beta}{\cos \alpha}$, where β is the angle made by the plane of section with the axis of the cone, and α is the semi-angle of the cone.

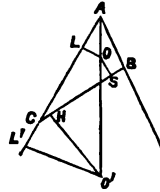
I. *Solution by J. J. WALKER, M.A.*

In the *Cambridge and Dublin Mathematical Journal*, Vol. vii., pp. 16, 17, I proved the more general property of any cone of the second degree. If a plane cut a cone in a conic, the planes of two subcontrary circular sections in two right lines, and the sphere containing those circular sections in a circle, the rectangle under the segments of any chord (or secant) of this circle drawn through any point on the conic is to the rectangle under perpendiculars drawn from the same point to the two right lines in the constant ratio of the products of the sines of the angles which the plane of the conic and any side of the cone respectively make with the planes of the circular sections.

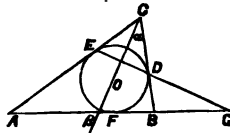
In the particular case of a section parallel to the line of intersection of two circular sections, it follows that $e^2 : 1 = \cos \beta \cos \beta' : \cos \alpha \cos \alpha'$; β, β' being the angles which the plane of the conic makes with normals to the circular sections of the cone; α, α' those which any side of the cone makes with the same normals. Finally, in the case of the right cone, the expression for the eccentricity given in the question plainly follows.

II. *Solution by Prof. WOLSTENHOLME; M. COLLINS, M.A.; and others.*

Let A be the vertex of the cone; BC the major axis of the section; O, O' the centres of the circles which touch BC in S, H which are the foci of the section; OL, O'L' perpendicular on AC; then CL = CS, and CL' = CH, therefore LL' = BC. But LL' = OO' cos α , and SH = OO' cos β , whence the eccentricity = $\frac{SH}{BC} = \frac{\cos \beta}{\cos \alpha}$.

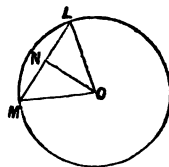
III. *Solution by Prof. TOWNSEND; G. EDMUNDSON; and others.*

The circle DEF, inscribed to the triangle ABC determined by the transverse axis AB of the section with the vertex C of the cone, having its centre O on the internal axis CO of the latter, and its point of contact F with AB at a focus F of the latter; its chord of contact DE with the remaining two sides BC and CA, which is of course perpendicular to CO, intersecting AB at its intersection G with the directrix corresponding to F; the eccentricity e of the section is consequently = AF : AG, or BF : BG, = AE : AG, or BD : BG, = sin AGE : sin AEG, or sin BGD : sin BDG, = cos β : cos α ; and therefore, &c.

IV. *Solution by the PROPOSER; A. M. NASH; and others.*

This theorem can be established in various ways; the following proof, when the section is an hyperbola, appears the simplest mode of arriving at the result.

Suppose a plane drawn through the vertex of the cone parallel to the plane of section, and describe any sphere with the vertex of the cone as centre, intersecting the cone in a lesser circle, and cutting the plane in the great circle LM. From O, the pole of the lesser circle, draw ON perpendicular to LM; then we have $\cos OL = \cos ON \cdot \cos LN$.



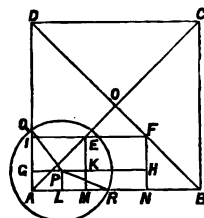
Again, by an elementary property, the eccentricity of the hyperbola is equal to the secant of half the angle between its asymptotes, that is (in the present example) equal to $\sec LN$; therefore $e = \sec LN = \frac{\cos ON}{\cos OL} = \frac{\cos \beta}{\cos \alpha}$.

By the principle of continuity, the property must hold for an elliptic section.

1321. (Proposed by the Editor.)—If two marbles are thrown at random on the floor of a rectangular room, whose length is a and breadth b , show (1) that the probability of their resting at a distance apart less than r is $\frac{r^2}{ab} \left\{ \pi - \frac{4}{3} \left(\frac{r}{a} + \frac{r}{b} \right) + \frac{1}{2} \frac{r^2}{ab} \right\}$, when r does not exceed b ; (2) extend the problem to the case when r is greater than b ; and (3) show therefrom that for a square of side a , the respective probabilities when r has the values $\frac{1}{4}a$, $\frac{1}{2}a$, a , $\frac{3}{2}a$, are $\frac{1}{16}(\pi - \frac{5}{3}) = .1566$, $\frac{1}{4}(\pi - \frac{2}{3}) = .4833$, $\pi - \frac{1}{2} = .5858$, and .9986.

I. Solution by STEPHEN WATSON.

Suppose the floor to be a square of side a as in the annexed figure, and $r = \frac{1}{4}a$; draw the lines IF, GH parallel to one side AB of the square, the former at a distance r from AB, cutting the diagonals in E, F. With P, a point in GH, as centre, and radius AI, draw a circle cutting AB, AD in R, Q.



Join PR, PQ, and draw the perpendiculars PL, EM, FN, the two latter cutting GH in K and H.

Put $AB = a$, $AI = PR = r$, $\angle PRL = \alpha$, $\angle PQG = \beta$, and $GP = x = r \sin \beta$.

When $\cos \beta > \sin \alpha$, or $\beta < \frac{1}{2}\pi - \alpha$, the circle cuts AB, AD each in one point only, and the area common to the square and circle is

$$S_1 = \frac{1}{2}r^2 \left(\frac{1}{2}\pi + \alpha + \beta + \sin \alpha \cos \alpha + \sin \beta \cos \beta + 2 \sin \alpha \sin \beta \right).$$

When $\beta > \frac{1}{2}\pi - \alpha$, the circle cuts AB, AD, each in two points, and the common area is $S_2 = r^2 (\alpha + \beta + \sin \alpha \cos \alpha + \sin \beta \cos \beta)$.

When P lies between K and H, the common area is

$$S_3 = r^2 \left(\frac{1}{2}\pi + \alpha + \sin \alpha \cos \alpha \right).$$

When P lies in the triangle OEF, the common area is πr^2 . Also the total number of positions of the two marbles is a^2 , and the rectangle IN and the triangle OEF comprise one-fourth of the square; hence if p be the chance that the distance apart of the marbles does not exceed r , we have

$$\begin{aligned} \frac{1}{4}a^2p &= \int_0^{1/r} d(r \sin \alpha) \left(\int_0^{1/r - r \sin \alpha} S_1 + \int_{1/r - r \sin \alpha}^{1/r} S_2 \right) d(r \sin \beta) \\ &\quad + \int_0^{1/r} d(r \sin \alpha) \int_r^{a-r} S_3 dx + \pi r^2 (\Delta OEF) \\ &= \frac{r^4}{6} \int_0^{1/r} (\sin^3 \alpha - 3 \sin \alpha - 2) \cos \alpha d\alpha \\ &\quad + r(a-r) \int_0^{1/r} S_3 \cos \alpha d\alpha + \frac{1}{4}\pi r^2 (a-2r) \\ &= \frac{r^2}{24} (6a^2\pi - 16ar + 3r^2); \end{aligned}$$

therefore

$$p = \frac{r^2}{a^2} \left(\pi - \frac{8}{3} \frac{r}{a} + \frac{1}{2} \frac{r^2}{a^2} \right).$$

II. Solution by W. S. B. WOOLHOUSE, F.R.A.S., F.S.C., &c.

Take the problem for a rectangle ABCD whose sides are AB = a , AD = b ; and let the limiting distance between the marbles (P, Q) = r , and the coordinates of P, AH = x , PH = y .

Then if we conceive a circle, radius r , to be drawn with P as a centre, the other marble Q, according to the proposed conditions, must lie in that portion of the rectangle which falls within this circle. But if lines EF, GH be drawn through P parallel to the sides, and dividing this particular surface into four parts, it will be evident that, when P takes all positions on the rectangle, each of these parts will pass through precisely the same values, so that a calculation with respect to one of them will be sufficient. If therefore S denote the surface of that part of the circle which falls on the rectangle PA, the number of positions conformable to the question will be represented by $4 \iint dx dy S$, the total number of possible positions being $ab \times ab = a^2b^2$. The sought probability is therefore

$$p = \frac{4}{a^2b^2} \iint dx dy S.$$

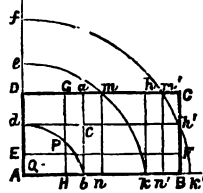
There are three distinct cases which it will be convenient to investigate separately.

Case I.—When the distance r , within which the marbles are required to lie, is less than b .

With radius r and centre A draw the quadrantal arc db , and through d , b draw dh' , ab parallel to the sides. Also to avoid the use of surds, when x and y are either of them less than r , let x' , y' denote complementary coordinates such that $x^2 + x'^2 = y^2 + y'^2 = r^2$.

First consider the positions of P on the rectangle Aa.

When y passes from 0 to x' , $S = xy$ and $\int dy S = \frac{1}{2}xx'^2$.



When $y = x' \dots r$, the portion of AH contained within the circle is y' ; therefore $\frac{dS}{dy} = y'$; and since $ydy = -y'dy'$, $\int ydS = -\frac{1}{2}y'^2$; therefore

$$\int dyS = yS - \int ydS = yS + \frac{1}{2}y'^2 \text{ between limits} = rS_r - x'(x') - \frac{1}{2}x'^2.$$

When $y = r \dots b$, S is constant, and $\int dyS = (b-r)S_r$.

Therefore, collecting these three values, we get, for the total limit

$$y=0 \dots b, \quad \int dyS = bS_r - \frac{1}{2}x'^2 - \frac{1}{2}x'^2 = bS_r - \frac{1}{2}x'^2 + \frac{1}{2}x'^2.$$

Integrating again for $x=0 \dots r$, $\iint dx dyS = b \int dxS_r - \frac{1}{2}x'^2$.

But $\frac{dS}{dx} = x'$, and, since $x dx = -x' dx'$, $\int x dS_r = -\frac{1}{2}x'^2$;

therefore $\int dxS_r = xS_r + \frac{1}{2}x'^2$ (between limits) $= rS_r - \frac{1}{2}x'^2$;

$$\text{therefore} \quad \iint dx dyS = brS_r - \frac{1}{2}bx'^2 - \frac{1}{2}x'^2 \dots (1).$$

Secondly, consider the positions of P on the rectangle BC.

When $y = 0 \dots r$, $\frac{dS}{dy} = y'$, and, as before,

$$\int dyS = yS + \frac{1}{2}y'^2 \text{ (between limits)} = rS_r - \frac{1}{2}r'^2.$$

When $y = r \dots b$, $S = S_r$ is constant, and $\int dyS = (b-r)S_r$.

Hence, for the total limits $y=0 \dots b$, $\int dyS = bS_r - \frac{1}{2}r'^2$;

$$\text{therefore} \quad \iint dx dyS = (a-r)(bS_r - \frac{1}{2}r'^2) \dots (2).$$

For the whole rectangle ABCD add (1) and (2); the result is

$$\iint dx dyS = abS_r - \frac{1}{2}(a+b)r'^2 + \frac{1}{2}r'^2, \text{ where } S_r = \frac{1}{2}r'^2;$$

$$\text{therefore} \quad p = \frac{r^2}{ab} \left\{ \pi - \frac{4}{3} \left(\frac{r}{a} + \frac{r}{b} \right) + \frac{1}{2} \frac{r^2}{ab} \right\}.$$

Case II.—When the distance r is greater than b and less than a .

With centre A and radius r draw the quadrantal arc cmk , and draw mn , kh parallel to DA.

First let P be on the rectangle Am; then $S = xy$,

$$\text{therefore} \quad \iint dx dyS = \frac{1}{2}x^2y^2 = \frac{1}{2}b^2b^2 \dots (1).$$

Secondly, let P be on the rectangle nh .

For $y = 0 \dots x'$, $S = xy$ and $\int dyS = \frac{1}{2}xx'^2$.

For $y = x' \dots b$, $\frac{dS}{dy} = y'$, and

$$\int dyS = yS + \frac{1}{2}y'^2 \text{ (between limits)} = bS_{x'} - x'(x') - \frac{1}{2}x'^2 + \frac{1}{2}b'^2.$$

Therefore for the total limits $y = 0 \dots b$,

$$\int dyS = bS_{x'} - \frac{1}{2}xx'^2 - \frac{1}{2}x'^2 + \frac{1}{2}b'^2 = bS - \frac{1}{2}r^2x + \frac{1}{2}x^2 + \frac{1}{2}b'^2.$$

And, integrating for $x = 0 \dots r$,

$$\iint dx dyS = b \int dxS_{x'} - \frac{1}{2}b'^2r + \frac{1}{2}b'^2r + \frac{1}{2}b'^2r^2 - \frac{1}{2}x'^2r^2.$$

But $\frac{dS_{x'}}{dx} = x'$, $\int x dS_{x'} = -\frac{1}{2}x'^2$; and

$$\int dx S_{ab} = x S_{ab} + \frac{1}{3} x^3 \text{ (between limits) } = r S_{ab} - b' (bb') - \frac{1}{3} b^3;$$

therefore $\iint dx dy S = b r S_r - b^2 b'^2 - \frac{1}{3} b^4 - \frac{2}{3} b'^4 + \frac{1}{3} b'^2 r^2 + \frac{1}{3} b'^2 r^2 - \frac{2}{3} r^4 \dots\dots (2).$

Lastly, when P is on the rectangle kC, $\frac{dS}{dy} = y'$,

therefore $\int dy S = y S + \frac{1}{2} y'^2 \text{ (} y = 0 \dots\dots b \text{)} = b S_r + \frac{1}{2} b'^2 - \frac{1}{2} r^2;$

therefore $\int dx dy S = (a-r) (b S_r + \frac{1}{2} b'^2 - \frac{1}{2} r^2) \dots\dots\dots (3).$

Adding (1), (2), (3), and substituting $r^2 - b^2$ for b'^2 , we get, for the whole rectangle ABCD, $\iint dx dy S = ab S_{ab} + \frac{1}{3} ab' r^2 - \frac{1}{3} ab^2 b' + \frac{1}{3} b^4 - \frac{1}{3} b^2 r^2 - \frac{1}{3} a r^2,$

where $S_{ab} = \frac{1}{3} \pi r^2 + \frac{1}{3} b b' - \frac{1}{3} r^2 \cos^{-1} \frac{b}{r};$

therefore $p = \frac{r^2}{ab} \left\{ \pi + \frac{4}{3} \frac{b'}{b} + \frac{2}{3} \frac{b b'}{r^2} - 2 \cos^{-1} \frac{b}{r} - \frac{4}{3} \frac{r}{b} \right\} - \frac{r^2}{a^2} + \frac{1}{6} \frac{b^2}{a^2}$

$$= \frac{r^2}{ab} \left\{ \pi + \phi \left(\frac{r}{b} \right) - \frac{4}{3} \frac{r}{b} \right\} - \frac{r^2}{a^2} + \frac{1}{6} \frac{b^2}{a^2},$$

where $\phi \left(\frac{r}{b} \right) = \frac{4}{3} \frac{b'}{b} + \frac{2}{3} \frac{b b'}{r^2} - 2 \cos^{-1} \frac{b}{r}$

is a small function, of the fifth order with respect to $\frac{b}{r}$, which admits of convenient tabulation. The marginal table is a brief specimen, and one a little more minute and extended would give the values accurately by inspection. If we put $b = r \cos \theta$, the function will take the following trigonometrical form $\phi (\sec \theta) = \frac{4}{3} \tan \theta + \frac{1}{3} \sin 2\theta - 2\theta$, from which a more extended table may be conveniently calculated.

Case III.—When the distance r is greater than a and less than $(a^2 + b^2)^{\frac{1}{2}}$.

First, supposing P to be on the rectangle Am', we have $S = xy$, and

$$\iint dx dy S = \frac{1}{2} b^2 b'^2 \dots\dots\dots (1).$$

Secondly, let P be on the rectangle n'C, and the integral with respect to y will be the same as in the rectangle n'h of Case II.; therefore

$$\int dy S = b S_{ab} - \frac{1}{2} r^2 x + \frac{1}{2} x^2 + \frac{1}{2} b'^2;$$

and, integrating for $x = b' \dots\dots a$,

$$\iint dx dy S = b \int dx S_{ab} + \frac{1}{2} a^4 - \frac{1}{2} b'^4 + \frac{1}{2} a b'^2 - \frac{1}{2} a^2 r^2 + \frac{1}{2} b'^2 r^2.$$

But $\frac{dS_{ab}}{dx} = x'$, and

$$\int dx S_{ab} = x S_{ab} + \frac{1}{2} x'^2 \text{ (between limits) } = a S_{ab} - b' (bb') - \frac{1}{2} b^3 + \frac{1}{2} a^3;$$

therefore $\iint dx dy S = ab S_{ab} + \frac{1}{2} (a^3 b + a b^3) + \frac{1}{2} a^4 - \frac{1}{2} b^4$

$$- b^2 b'^2 - \frac{1}{2} b'^4 - \frac{1}{2} a^2 r^2 + \frac{1}{2} b'^2 r^2 \dots\dots\dots (2).$$

$\frac{r}{b}$	$\phi \left(\frac{r}{b} \right)$	diff.
1.0	.0000	41
1.1	.0041	161
1.2	.0202	287
1.3	.0489	404
1.4	.0893	506
1.5	.1399	594
1.6	.1993	670
1.7	.2663	737
1.8	.3400	793
1.9	.4193	844
2.0	.5037	

Adding (1) and (2), and putting $r^2 - b^2$ for b'^2 , we get for the whole rectangle ABCD,

$$\iint dx dy S = abS_{ab} + \frac{1}{2} (a^2b + ab^2) + \frac{1}{24} (a^4 + b^4) - \frac{1}{2} (a^2 + b^2) r^2 - \frac{1}{2} r^4,$$

where $S_{ab} = \frac{1}{2} \pi r^2 + \frac{1}{2} (aa' + bb') - \frac{1}{2} r^2 \left(\cos^{-1} \frac{a}{r} + \cos^{-1} \frac{b}{r} \right),$

also $\frac{1}{2} (a^2b + ab^2) = \frac{1}{2} (a'b + ab') r^2 - \frac{1}{2} ab (aa' + bb');$

$$\begin{aligned} \therefore p &= \frac{r^2}{ab} \left\{ \pi + \frac{4}{3} \left(\frac{a'}{a} + \frac{b'}{b} \right) + \frac{2}{3} \frac{aa' + bb'}{r^2} - 2 \left(\cos^{-1} \frac{a}{r} + \cos^{-1} \frac{b}{r} \right) - \frac{1}{2} \frac{r^2}{ab} \right\} \\ &\quad - \left(\frac{r^2}{a^2} + \frac{r^2}{b^2} \right) + \frac{1}{6} \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} \right) \\ &= \frac{r^2}{ab} \left\{ \pi + \phi \left(\frac{r}{a} \right) + \phi \left(\frac{r}{b} \right) - \frac{1}{2} \frac{r^2}{ab} \right\} - \left(\frac{r^2}{a^2} + \frac{r^2}{b^2} \right) + \frac{1}{6} \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} \right), \end{aligned}$$

in which ϕ denotes the same form of function as in Case II.

When the proposed figure is a square, then $a=b$, and Case II. is not needed.

When $r < a$,
$$p = \frac{r^2}{a^2} \left(\pi - \frac{8}{3} \frac{r}{a} + \frac{1}{2} \frac{r^2}{a^2} \right).$$

When $r > a$,
$$p = \frac{r^2}{a^2} \left\{ \pi + 2\phi \left(\frac{r}{a} \right) - \frac{1}{2} \frac{r^2}{a^2} \right\} - \frac{2r^2}{a^2} + \frac{1}{3}$$

[A Solution by Professor CROFTON is given on p. 84 of Vol. XIV. of the *Reprint*.]

3638. (Proposed by R. W. GENESE, B.A.)—Prove geometrically that the straight line joining the middle points of BC and AD, (D being the point where the inscribed circle touches BC,) passes through the centre of the inscribed circle.

I. *Solution by S. WATSON; A. MARTIN; and others.*

Let E, F be middle points of BC, AD; and draw AH, FG, DO perpendicular to BC, the last meeting EF in O. Then DG=GH, $ED = \frac{1}{2}(AB-AC)$; therefore $BC \cdot EH = \frac{1}{2}(AB^2-AC^2) = ED(AB+AC)$, and $EG \cdot BC = \frac{1}{2}(ED+EH) BC$

$$= \frac{1}{2}(AB+AC+BC) ED;$$

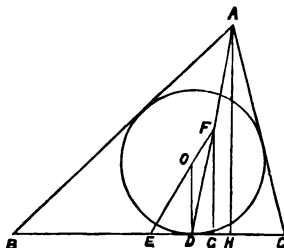
hence we have

$$ED : DO = EG : GF = EG \cdot BC : GF \cdot BC$$

$$= \frac{1}{2}(AB+BC+CA) ED : GF \cdot BC$$

$$\text{therefore } \frac{1}{2}(AB+BC+CA) OD$$

$$= GF \cdot BC = \text{triangle ABC};$$



therefore $OD = r$, which proves the theorem in the question, which is a particular case of the celebrated problem of the locus of the centre of conics inscribed in a quadrilateral.

II. Solution by the PROPOSER.

Let D' be the point where an escribed circle touches BC , [the additions to the figure may be easily imagined] then we know that $BD' = DC$. Thus if E be the middle point of BC , it is also of $D'D$. Again let AD' meet the circle in P ; then because A is the centre of similitude of the circles, the tangent at P is parallel to BC ; therefore PD is a diameter of the inscribed circle. But if F be the middle point of AD , EF bisects DP , and therefore passes through centre of the circle.

The quickest proof, however, is by mechanics. Thus take forces represented in position and magnitude by BA , CA , BD , CD . The resultant of BA , BD passes through F , as does that of CA , CD ; *i.e.*, the resultant of the system passes through F . Again the resultant of BA and CA passes through E ; and BD , CD pass through E ; therefore the resultant passes through E ; *i.e.*, FE is the line of action of the resultant.

But again [if Q , R be the points of contact of the inscribed circle on AC , AB] the parts BD , BR ; CD , CQ ; RA , QA have their resultants passing through centre O of the inscribed circle (or conic); this centre therefore lies on EF .

The above proof with the *same* letters applies if DEF be either of the escribed circles on the angles B or C .

3842. (Proposed by T. MITCHESON, B.A.)—Two radii vectores, r and r_1 , are drawn from the vertex of a parabola at right angles; show that the inclination of one of them to the axis of x is $\cot^{-1} r^{\frac{1}{2}} r_1^{-\frac{1}{2}}$.

*Solution by A. B. EVANS, M.A.; Q. W. KING; J. H. SLADE;
the PROPOSER; and others.*

Let θ and $\phi = \frac{1}{2}\pi - \theta$ be the angles made by r and r_1 with the axis of x ;
then $x = r \cos \theta$, $x_1 = r_1 \cos \phi$, $y^2 = r^2 \sin^2 \theta$, $y_1^2 = r_1^2 \sin^2 \phi$,
also $r \cos \theta : r_1 \sin \theta = r^2 \sin^2 \theta : r_1^2 \cos^2 \theta$,
hence we have $r = r_1 \cot^2 \theta$, whence $\theta = \cot^{-1} r^{\frac{1}{2}} r_1^{-\frac{1}{2}}$.

1279. (Proposed by the EDITOR.)—From a point O in the plane of a triangle ABC , of which the area is Δ , the centre of the circumscribed circle P , and its radius R , draw perpendiculars OD , OE , OF to the sides BC , CA , AB ; and let Δ_1 be the area of the triangle DEF , and $OP = k$; then prove (1) that $\Delta_1 = \frac{1}{4} \left(1 - \frac{k^2}{R^2}\right) \Delta$; and hence show (2) that when Δ_1 is constant the locus of O is a circle of centre P and radius k ; (3) that the

maximum value of Δ_1 is $\frac{1}{4}\Delta$ when O coincides with P ; and (4) that Δ_1 degenerates into a straight line when O is on the circumscribed circle.

Solution by J. McDOWELL, M.A.

Draw the three perpendiculars of the given triangle, AM , BN , CL . Produce BO to meet the circumference in G ; join AG , and draw OH perpendicular to AG , and DK perpendicular to EF .

Then we have

$$DK \cdot FE : GO \cdot OB = \left\{ \begin{array}{l} FE : OR \\ DK : GO \end{array} \right\}.$$

But $FE : OB = CL : CB$ (as is shown below) and $DK : GO = \left\{ \begin{array}{l} DK : OH \\ OH : GO \end{array} \right\}$.

Again, $DK : OH = DF : AO$ (as is shown below) $= BN : AB$,
and $OH : GO = AM : AC$.

Hence we have $DK \cdot FE : GO \cdot OB = \left\{ \begin{array}{l} CL : CB \\ BN : BA \\ AM : AC \end{array} \right\}$.

[But $DK \cdot FE = 2\Delta_1$, and $GO \cdot OB = R^2 - k^2$; also $BC \cdot CA \cdot AB = 4R\Delta$, and $AM \cdot BN \cdot CL = \frac{2}{R}^2$ (*Diary* for 1843, p. 93), therefore $2\Delta_1 : R^2 - k^2$

$$= \Delta : 2R^2, \text{ therefore } \Delta_1 : \Delta = R^2 - k^2 : 4R^2, \text{ or } \Delta_1 = \frac{1}{4} \left(1 - \frac{k^2}{R^2} \right) \Delta.]$$

When, therefore, the area (Δ_1) of the triangle DEF is constant, k must be constant, and the locus of O will be a circle of radius k concentric with that drawn round the given triangle ABC . Moreover, DEF will evidently be a maximum when $k=0$, that is when O is at the centre of the circumscribed circle of the triangle ABC ; and since D, E, F are then the middle points of the sides BC, CA, AB , the maximum area of the triangle DEF will be one-fourth of that of the given triangle.

When $k=R$, that is, when O is on the circumscribed circle, we have $\Delta_1=0$, which shows that the feet of the perpendiculars on the sides from a point on the circumscribed circle are in a straight line, a well-known theorem.

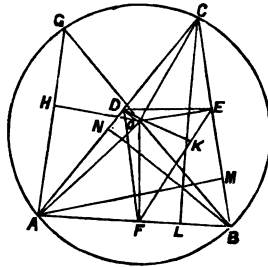
NOTE.—It is not difficult to prove geometrically that the locus of a point such that if the squares of its distances from n given points be multiplied by given constants, positive or negative, and added, their sum shall be constant, is a circle. By the help of this theorem, the property in this Question may be readily extended to the case of any polygon.

[We add here proofs of two properties assumed in Mr. McDOWELL's investigation:—

The circle on OB as diameter will pass through E, F ; and if the diameter of this circle through E meet its circumference in Q , then the angle $EQF = EBF$. Hence the triangles EQF, CBL are similar, therefore

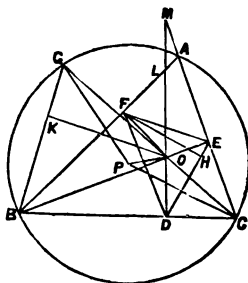
$$FE : OB \text{ (or } EQ) = CL : CB.$$

Again, the angle $OFD = OAD$, and $OFE = OBE = CAG$, hence the angle $DFK = OAH$, and therefore $DK : OH = DF : AO.]$



II. Solution by MATTHEW COLLINS, B.A.

Let ABC (in the figure below) be the given triangle, and OD, OE, OF the perpendiculars. Produce CO to meet the circumscribed circle in G ; join BG , on which draw the perpendicular OK ; and from F draw FH perpendicular to DE . Let DO produced meet AB in L , and AC in M ; then the pairs of triangles LOB and LFD , MOC and MED , OBK and FDH are similar; therefore $FH : OK = DF : OB = LD : LB$ is constant, since the triangle LBD is given in species; also $DE : OC = MD : MC$ is constant, since the triangle MDC is given in species. Again, the angle OGK is equal to the constant angle A , therefore the triangle OGK is given in species, and therefore $OK : OG$ is constant. But $FH : OK$ has been proved constant, therefore $FH : OG$ is constant; and $DE : OC$ has also been proved constant, therefore $DE.FH : OC.OG$ is constant. But $DE.FH = 2\Delta DEF$, which is to be constant (by hypothesis), therefore $OC.OG$ must be so too. Now, $OC.OG = PG^2 - PO^2$, P being the centre of the circumscribed circle; and as the radius PG is invariable, therefore PO is constant, and its end P is fixed, therefore the required locus of O is a circle about the centre P ; it is therefore concentric with the circle described about the triangle ABC .



Since the triangle DEF has a constant ratio to $PG^2 - PO^2$, whatever be the position of the point O , therefore when triangle DEF is a *maximum*, PO must be a *minimum* (i.e. = zero), and then O will coincide with P . The points D, E, F will then be the middle points of the sides of the given triangle ABC ; hence the *maximum* value of triangle DEF is one-fourth of the original triangle ABC .

NOTE.—If the circle and the point O remain fixed, and the triangle ABC be turned about within the circle, the triangle formed by joining the feet of perpendiculars from O on the sides, remains constant, since its ratio to $CO.OG$ depends solely on the magnitude of the angles of the triangle ABC , as appears from the foregoing demonstration.

[If we put $OD = \alpha$, $OE = \beta$, $OF = \gamma$, the equation of the locus of O when Δ_1 is constant is at once seen to be, in trilinear coordinates (α, β, γ) ,

$$\beta\gamma \sin A + \gamma\alpha \sin B + \alpha\beta \sin C = 2\Delta_1,$$

$$\text{or} \quad a\beta\gamma + b\gamma\alpha + c\alpha\beta - 4R\Delta_1 = \frac{R\Delta_1}{\Delta^2} (a\alpha + b\beta + c\gamma)^2,$$

which shows that the locus is a circle about P as centre.]

2104. (Proposed by WILLIAM CURTIS OTTER, F.R.A.S.)—Required a full investigation of the following astronomical problem:—(1) by properties of the cone, (2) independently of the cone. To find the locus of the extremity of the shadow of a vertical gnomon, erected on a horizontal plane, on a given day, in a given latitude.

Solution by the EDITOR.

1. *First Solution, "by properties of the cone."* As a purely geometrical problem, the Question may be enunciated as follows. A straight line passes through a given point, and moves round at a constant inclination to a fixed line, required the locus of its intersection with a given plane. The straight line is the direction of the sun's rays, the top of the gnomon is the given point, a parallel through it to the axis of the earth is the fixed line, the sun's polar distance is the constant inclination, and the horizon is the given plane. Now the straight line will generate a conical surface, and the intersection of this surface with a plane is a conic section, which will be a parabola, an ellipse, or an hyperbola, according as the semi-vertical angle of the cone is equal to, less than, or greater than the inclination of its axis to the cutting plane. But the sun's polar distance is the semi-vertical angle of the cone, and the inclination of its axis to the horizontal (cutting) plane is the altitude of the elevated pole, or the latitude of the gnomon; hence the locus of the end of the gnomon's shadow will be a parabola, an ellipse, or an hyperbola, according as the sun's polar distance is equal to, less than, or greater than the latitude of the gnomon.

The vertex of the conic is in the meridian line, and it will be at the foot of the gnomon when the sun passes through the zenith. The locus will be convex or concave towards the pole according as the sun and the zenith are on the same or on opposite sides of the equator. When the sun is in the equator, the locus is a straight line; for then the cone becomes a plane, which intersects the horizon in a straight line (the required locus) perpendicular to the meridian line. At the poles the locus is a circle, for there the horizon is perpendicular to the axis of the cone; at places within the polar circles, the locus will be all the varieties of a conic, at different times of the year; but at all other places the locus must always be an hyperbola (or a straight line), since the sun's polar distance will be greater than the latitude of the gnomon.

2. *Second Solution, "independently of the cone."* Let α , β , δ , be the sun's azimuth (estimated from the south), altitude, and polar distance, λ the latitude of the gnomon, h its height, and x , y the rectangular co-ordinates of the end of its shadow, referred to its foot as origin, and the meridian line towards the pole as positive axis of x . Then from the spherical triangle whose angular points are the pole (P), the sun (S), and the zenith

$$\begin{aligned} \cos PS &= \cos PZ \cos ZS - \sin PZ \sin ZS \cos SZP, \\ \text{or} \quad \cos \delta &= \sin \beta \sin \lambda + \cos \alpha \cos \beta \cos \lambda. \end{aligned}$$

$$\text{But } \cos \alpha = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}, \quad \sin \beta = \frac{h}{(x^2 + y^2 + h^2)^{\frac{1}{2}}}, \quad \cos \beta = \left(\frac{x^2 + y^2}{x^2 + y^2 + h^2} \right)^{\frac{1}{2}};$$

hence the equation of the required locus is

$$(x^2 + y^2 + h^2) \cos^2 \delta = (h \sin \lambda - x \cos \lambda)^2;$$

or putting $\mu = \pi - (\lambda + \delta)$ = the sun's meridian altitude, and $\nu = (\lambda - \delta)$, the equation becomes

$$y^2 \cos^2 \delta + x^2 \sin \mu \sin \nu + hx \sin 2\lambda = h^2 \cos \mu \cos \nu.$$

The second solution gives the same results as the first. For since the equation of the locus is of the second degree, the locus is a conic, which will be a parabola, an ellipse, or an hyperbola, according as $\sin \nu = > < 0$, or according as $\delta = < > \lambda$. Again, at the poles, where $\lambda = 90^\circ$, the locus becomes the circle $x^2 + y^2 = h^2 \tan^2 \delta$; and at the equinoxes, when $\delta = 90^\circ$, the locus is the straight line $x = h \tan \lambda$.

3952. (Proposed by T. MITCHESON, R.A.)—A is a point on the circumference of a given circle, P a point without it; AP cuts the circle in B, and $AP = n \cdot AB$; show that the locus of P is a circle touching the given circle in A.

Solution by the Hon. ALGERNON BOURKE; and many others.

Produce AD, the diameter of the given circle, to E, so that $AE = n \cdot AD$, and join DB, EP; then, by the similar triangles APE, ABD, we see that P is a right angle; therefore the locus of P must be the circle of which AE is the diameter; and since the line joining their centres passes through A, they must touch at A.

2107. (Proposed by N'IMPORTE.)—A certain territory is bounded by two meridian circles, and by two parallels of latitude, which differ in longitude and latitude respectively by one degree, and is known to lie between the limits α and β of latitude; show that, if r be the radius of the earth, the mean superficial area is $\frac{\pi r^2 \sin \frac{1}{2} \cos \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\beta - \alpha - 1)}{45 (\beta - \alpha - 1)}$.

Solution by STEPHEN WATSON.

Let θ be the latitude of a place equally distant from the two parallels of latitude; then the two latitudes are $\theta - \frac{1}{2}$, $\theta + \frac{1}{2}$, and the surface lying between these parallels is

$$2\pi r^2 \left\{ \sin \left(\theta + \frac{1}{2} \right) - \sin \left(\theta - \frac{1}{2} \right) \right\} = 4\pi r^2 \sin \frac{1}{2} \cos \theta,$$

r being the radius of the earth, also the portion of this surface lying between the meridian circles is $\frac{1}{\pi^2} \pi r^2 \sin \frac{1}{2} \cos \theta$.

Hence if α, β be the two latitudes between which the territory is known to lie, the required area is

$$\frac{1}{90} \pi r^2 \sin \frac{1}{2} \int_{\alpha+\frac{1}{2}}^{\beta-\frac{1}{2}} \cos \theta \frac{d(r\theta)}{r(\beta-\alpha-1)} = \frac{\pi r^2 \sin \frac{1}{2}}{90(\beta-\alpha-1)} \left\{ \sin(\beta - \frac{1}{2}) - \sin(\alpha + \frac{1}{2}) \right\},$$

which reduces to the result given in the Question.

3911. (Proposed by C. H. HINTON.)—Show that $\frac{1}{3} n(2n-1)(n+5) + I(\frac{1}{3}n) = M(3)$.

Solution by the PROPOSER.

$$\begin{aligned} \frac{1}{3} n(2n-1)(n+5) &= \frac{1}{3} (2n^3 + 3n^2 - 5n) = \frac{1}{3} (2n^3 + 3n^2 + n) - n \\ &= \frac{1}{3} n(n+1)(2n+1) - n = (1^2 - 1) + (2^2 - 1) + (3^2 - 1) + \dots + (n^2 - 1). \end{aligned}$$

Adding the $I(\frac{1}{3}n)$ is equivalent to cancelling the (-1) after every multiple of 3 in the above, whence, by FERMAT'S theorem, every term is a multiple of 3. Hence the original expression is a multiple of 3.

Similarly, we may show that

$$\frac{1}{30} (6n^5 + 15n^4 + 10n^3 - 31n) + I(\frac{1}{3}n) = M(5), \text{ \&c.}$$

1059. (Proposed by the EDITOR.)—The two straight arms of a bent lever are of the lengths a, ka (where $k > 1$) and contain an obtuse angle the cosine of whose supplement is λ ; show that if from the ends of each of the arms a piece be cut at random, and the lever be then placed on a fulcrum at the angle, the probability that, in the position of equilibrium, one of the arms will rise above the fulcrum, is

$$\frac{\lambda(1+k^2)}{2k} \text{ or } \frac{\lambda^2 + 2\lambda k - 1}{2\lambda k},$$

according as the shorter arm does or does not rest above the fulcrum when the lever is suspended there before the cutting.

Solution by STEPHEN WATSON.

Let AB, AC be the arms, of which AC is the less; BP, CQ the pieces cut off. Bisect AP, AQ in M, N; divide AM in D, so that AB : AQ = AD : DM, and draw DG parallel to AQ, meeting MN in G; then G is the centre of gravity of AP, AQ. Join AG, and produce BA to E. Put AB = a , AC = b , AP = x , AQ = y , $\angle CAE = \alpha$, and $\cos \alpha = \lambda$; then we easily

$$\text{find } AD = \frac{x^2}{2(x+y)}, \text{ and } DG = \frac{y^2}{2(x+y)};$$

hence the rectangular coordinates of G, measured

$$\text{from A, are } x_1 = \frac{x^2 - y^2 \cos \alpha}{2(x+y)}, \quad y_1 = \frac{y^2 \sin \alpha}{2(x+y)}.$$

Now in order that the arm AQ may rise above the horizontal, the angle GAC must be $> 90^\circ$; but when $\angle GAC = 90^\circ$,

$$\frac{\sin \alpha}{\cos \alpha} = \frac{x_1}{y_1} = \frac{x^2 - y^2 \cos \alpha}{y^2 \sin \alpha}, \text{ therefore } y = x \cos^{\frac{1}{2}} \alpha.$$

At this stage the question takes two forms, according as $b > a \cos^{\frac{1}{2}} \alpha$.

First, let $b > a \cos^{\frac{1}{2}} \alpha$. In this case the limits are $x = a$, $x = y \sec^{\frac{1}{2}} \alpha$; $y = 0$, $y = a \cos^{\frac{1}{2}} \alpha$; hence the chance of AQ rising above the horizontal is

$$\frac{1}{ab} \iint dy dx = \frac{1}{ab \cos^{\frac{1}{2}} \alpha} \int_0^{a \cos^{\frac{1}{2}} \alpha} (a \cos^{\frac{1}{2}} \alpha - y) dy = \frac{a \cos^{\frac{1}{2}} \alpha}{2b} \dots \dots \dots (1).$$

In like manner the chance that AP may rise above the horizontal is $\frac{b \cos^{\frac{1}{2}} \alpha}{2a}$; hence the required chance in this case is

$$\frac{(a^2 + b^2) \cos^{\frac{1}{2}} \alpha}{2ab} = \frac{\lambda(1+k^2)}{2k}.$$

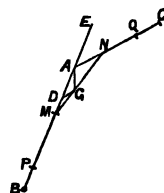
But when $b < a \cos^{\frac{1}{2}} \alpha$, the limits of y are $y = 0$, $y = b$, therefore (1) becomes

$$\frac{1}{ab \cos^{\frac{1}{2}} \alpha} \int_0^b (a \cos^{\frac{1}{2}} \alpha - y) dy = 1 - \frac{b}{2a \cos^{\frac{1}{2}} \alpha};$$

hence the chance in this case is

$$1 - \frac{b}{2a \cos^{\frac{1}{2}} \alpha} + \frac{b \cos^{\frac{1}{2}} \alpha}{2a} = 1 - \frac{b(1 - \cos \alpha)}{2a \cos^{\frac{1}{2}} \alpha} = \frac{\lambda^2 + 2\lambda k - 1}{2\lambda k}.$$

[The EDITOR'S Solution is given on p. 74 of Vol. IV. of the *Reprint*.]



3.35. (Proposed by S. WATSON.)—Find the average length of the radius of curvature in an ellipse.

Solution by the PROPOSER.

Let θ be the eccentric angle of the point where the normal is drawn; then an element of the curve at that point is $-(a^2 \sin^2 \theta + b^2 \cos^2 \theta) d\theta$,

and the radius of curvature is $\frac{(a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{3}{2}}}{ab}$;

hence, if l be the length of a quadrantal arc of the ellipse, the average required is

$$-\frac{1}{abl} \int_{\frac{3\pi}{2}}^0 (a^2 \sin^2 \theta + b^2 \cos^2 \theta)^{\frac{3}{2}} d\theta = \frac{\pi}{16abl} (3a^4 + 2a^2b^2 + 3b^4).$$

When $b=a$, $l=\frac{1}{2}a\pi$, and this becomes $=a$, as it ought to be.

[Mr. MARTIN and others suppose the abscissas of the points at which the normals are drawn to increase uniformly; and, upon this supposition, the required average will be

$$\frac{1}{a} \int_0^a \frac{(a^2 - e^2 x^2)^{\frac{3}{2}}}{ab} dx = \frac{1}{2}a \left\{ 5 - 2e^2 + \frac{3 \sin^{-1} e}{(1-e^2)^{\frac{1}{2}}} \right\};$$

and when $a=b$ or $e=0$, this becomes $\frac{1}{2}a(5+3)=a$.]

2109. (Proposed by N'IMPORTE.)—A regular hexagon is composed of six equal heavy rods each of weight w moveable about their angular points, and two opposite angles are connected by a horizontal string; one rod is placed on a horizontal plane, and a weight W is placed at the middle point of the highest rod; show that the tension of the string is

$$\frac{1}{3} \sqrt{3} (W + 3w).$$

Solution by the PROPOSER.

Suppose the highest rod moved upwards through a very small space, so that the inclination of each of the oblique rods to the horizon from being α ($=60^\circ$) becomes $\alpha + \delta\alpha$. Then, if unity represent the length of each rod, the points of application of the string are each displaced through a horizontal distance $\sin \alpha \cdot \delta\alpha$, towards the centre; the centre of gravity of the upper five rods is raised through a vertical distance $\frac{5}{2} \cos \alpha \cdot \delta\alpha$; and the upper weight through a vertical distance $2 \cos \alpha \cdot \delta\alpha$. Therefore, by the so-called *principle of virtual velocities*, if T be the tension of the string,

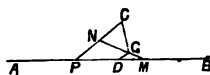
we have $5w \cdot \frac{5}{2} \cos \alpha \cdot \delta\alpha + W \cdot 2 \cos \alpha \cdot \delta\alpha = 2T \cdot \sin \alpha \cdot \delta\alpha$;

therefore $T = (W + 3w) \cot \alpha = \frac{W + 3w}{\sqrt{3}}.$

1058. (Proposed by the Editor.)—A straight piece of wire is bent at random into two arms, and suspended by an extremity. Show that the chance that the angle will, in the position of equilibrium, rise above the point of suspension, is $\frac{1}{2} - \frac{\sqrt{2}}{\pi} \log(\sqrt{2} + 1)$, or $\cdot 10325 = \frac{\pi}{24}$ nearly.

Solution by STEPHEN WATSON.

Let AB be the piece of wire; PB, PC the arms after it is bent, which bisect in M, N. Take PD : DM = PB : PC, and draw DG parallel to PC, cutting MN in G. Then G is the common centre of gravity of PB, PC. Join CG; put AB = a, AP = PC = x, the angle CPB = θ , and let $(x_1, y_1), (x_2, y_2)$ be the rectangular coordinates of C, G, measured from P.



Now we easily find $PD = \frac{(a-x)^2}{2a}$, $DG = \frac{x^2}{2a}$;

hence we find $x_1 = x \cos \theta$, $y_1 = x \sin \theta$,

also $x_2 = \frac{(a-x)^2 + x^2 \cos \theta}{2a}$, and $y_2 = \frac{x^2 \sin \theta}{2a}$.

When suspended by the point C, CG must be vertical; therefore, in order that P may be above C, the angle PCG must be greater than a right angle, but when PCG is equal to a right angle

$$\frac{\sin \theta}{\cos \theta} = -\frac{x_1 - x_2}{y_1 - y_2} = \frac{(x^2 - 2ax) \cos \theta + (a-x)^2}{(2ax - x^2) \sin \theta},$$

therefore $(a-x)^2 \cos \theta = 2ax - x^2$, or $\theta = \cos^{-1} \frac{2ax - x^2}{(a-x)^2}$;

consequently the chance of P being above C is

$$\frac{1}{\pi} \cos^{-1} \frac{2ax - x^2}{(a-x)^2};$$

also the chance of the angle being indefinitely near to P is $\frac{dx}{a}$; moreover,

the conditions, that it may be suspended from either C or B, and that x may be measured from either end of the rod, counterbalance each other; hence the required chance is

$$\frac{1}{a\pi} \int \cos^{-1} \frac{2ax - x^2}{(a-x)^2} dx = \frac{x}{a\pi} \cos^{-1} \frac{2ax - x^2}{(a-x)^2} - \pi \left(2 \sec^{-1} \frac{(a-x)\sqrt{2}}{a} + \sqrt{2} \log_e \left\{ a - x - [(a-x)^2 - \frac{1}{2}a^2]^{\frac{1}{2}} \right\} \right);$$

and, taking this integral between the limits $x=0$, $x=a(1-\frac{1}{\sqrt{2}})$, the latter of which makes θ vanish, we find the required probability to be

$$\frac{1}{2} - \frac{\sqrt{2}}{\pi} \log_e (\sqrt{2} + 1).$$

[The Editor's Solution is given on p. 17 of Vol. III. of the *Reprint*.]

1264. (Proposed by GEOMETRICUS.)—On donne un angle ABC , un point M et une longueur a , et on demande de faire passer par le point M une droite telle que la partie XY interceptée entre les côtés de l'angle soit égale à la droite donnée a .

Solution by the EDITOR.

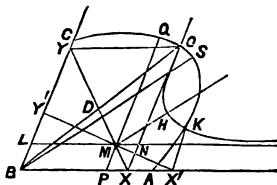
1. Suppose XY to be the required line; construct the parallelograms $BXOY$, $BPML$, and let the diagonals of the former meet in D .

Then, when XY passes through the fixed point M , the parallelogram OM is equal (Euc. I., 43,) to the constant parallelogram MB ; and therefore the locus of O (the corner opposite to B of the parallelogram on XY as diagonal) is a given hyperbola along the asymptotes LMN , PMQ .

Again, if a straight line XY of constant length moves with its ends on two fixed straight lines AB , BC , the locus of its middle point D (and therefore also, the locus of O) is, by a well-known property (see the Solution of Quest. 1600, *Reprint*, Vol. III., p. 55,) an ellipse having B for its centre, and its major axis along the bisector of the angle ABC .

When, therefore, XY both passes through a fixed point M , and is of given length a , as in the Question, the positions of O , and thence of XY , will be given by the intersections of an ellipse and an hyperbola. Hence we obtain the following construction:—Make $BA = BC = a$; on the bisector of the angle ABC take BS such that SZ , perpendicular to BS and meeting BA in Z , may be equal to a ; also on the bisector of the angle NMQ take MH such that the rhombus on MH as diagonal, and with its sides along MN , MQ , may be equal to the parallelogram MB . With S as vertex, and BS as major semi-axis, and MN , MQ as asymptotes, draw the ellipse ASC ; and with H as vertex, HM as major semi-axis, and MN , MQ as asymptotes, draw the hyperbola OHK , intersecting the ellipse ASC in O and K ; through O , K draw OX , KX' parallel to BC , and from X , X' draw through M the two lines XMY , $X'MY'$; then either of these will, by what is given above, satisfy the conditions of the problem.

It is evident that the other branch of the hyperbola will cut the ellipse in two points which will determine two other positions of the required line; but in each of these latter positions, one of its ends will be on AB or BC , and the other on the prolongations of these lines through B . Wherever the point may be within the angle ABC , the latter two positions of XY will always be possible, but the former not always. In fact it is easy to assign an area-locus within which M must fall between the lines AB , BC , in order that it may be possible to draw through it a line which shall have the part intercepted between AB , BC , and not between their continuations, equal to the given length a . If ABC be a right angle, the area-locus will be the quadrant of a four-branched hypocycloid within ABC ; and when ABC is oblique, it will be a sector of a curve of similar form, the bounding curve, in both cases, being the envelope of a line of length a , which moves with its ends on AB , BC . Hence, if two indefinitely extended lines and a point in their plane be given, it will be possible to draw through the point always two, sometimes four, but never more than four, straight lines such that the part intercepted between the fixed lines shall be of a given length.



2. Let $BP = b$, $PM = c$, $\cos B = \lambda$, $BX = x$, $BY = y$; then the passing of XY through the fixed point M , (or, what is equivalent thereto, the constant area of the parallelogram MO , which is always equal to MB), and the constant length of XY , may be expressed by the following equations:—

$$(x-b)(y-c) = bc, \quad x^2 + y^2 - 2\lambda xy = a^2 \dots\dots\dots (1, 2).$$

Eliminating xy from (1), (2), the resulting equation may be put into the form

$$(x-\lambda c)^2 + (y-\lambda b)^2 = a^2 + \lambda^2(b^2 + c^2) \dots\dots\dots (3).$$

Now x and y may be determined geometrically from the intersection of (1) and (3); for referring these variables to rectangular axes, (3) will represent a circle, the coordinates of whose centre are $(\lambda c, \lambda b)$ and the square of the radius $a^2 + \lambda^2(b^2 + c^2)$, and (1) will represent a rectangular hyperbola, having its asymptotes parallel to the axes of reference, the coordinates of its centre (b, c) and the square of its semi-axis $2bc$.

If we eliminate y from (1), (2) we obtain

$$\frac{a^2 - x^2}{x^2} = \frac{c^2}{(x-b)^2} - \frac{2\lambda c}{x-b},$$

$$\text{or} \quad x^4 - 2(b + \lambda c)x^3 + (b^2 + c^2 - a^2 + 2\lambda bc)x^2 + 2a^2bx - a^2b^2 = 0 \dots\dots (4).$$

Giving x the values $-\infty, 0, b, +\infty$, the left hand side of (4) gives results which are $+, -, +, +$ respectively; hence this equation has one real negative root, one real positive root between 0 and b , and two real or imaginary roots between b and ∞ . Thus one position of the required line always cuts BA to the left of B , another between B and P , and sometimes two, and sometimes none to the right of P .

3891. (Proposed by Dr. HART.)—Find n numbers whose sum is a square, and the sum of their squares a biquadrate.

Solution by SAMUEL BILLS.

Let $u_1, u_2, u_3 \dots u_n$ denote the n numbers; we have to find

$$u_1 + u_2 + u_3 + \dots + u_n = \square, \quad u_1^2 + u_2^2 + u_3^2 + \dots + u_n^2 = \text{biquadrate} \dots (1, 2).$$

First, to make (2) a square; assume

$$u_1 = v_1^2 + v_2^2 + v_3^2 \dots + v_{n-1}^2 - v_n, \quad u_2 = 2v_n v_1, \quad u_3 = 2v_n v_2, \quad \dots, \quad u_n = 2v_n v_{n-1};$$

then we shall have (2) $= (v_1^2 + v_2^2 + v_3^2 + \dots + v_n^2)^2$.

We must now have $v_1^2 + v_2^2 + \dots + v_n^2 = \square$.

$$\text{Assume} \quad v_1 = x_1^2 + x_2^2 + x_3^2 + \dots + x_{n-1}^2 - x_n, \quad v_2 = 2x_n x_1,$$

$$v_3 = 2x_n x_2, \quad v_4 = 2x_n x_3, \quad v_n = 2x_n x_{n-1};$$

then we shall have $u_1^2 + u_2^2 + \dots + u_n^2 = (x_1^2 + x_2^2 + \dots + x_n^2)^4$.

It now remains to satisfy (1).

From what has been done we readily find (putting r for $x_1 + x_2 + x_3 \dots + x_{n-2}$, and s for $x_1^2 + x_2^2 + x_3^2 \dots + x_{n-1}^2$),

$$u_1 = (s + x_n^2) - 8x_n^2 x_{n-1}, \quad u_2 = 4x_n x_{n-1} (s - x_n^2),$$

$$u_3 = 8x_n^2 x_{n-1} x_1, \quad u_4 = 8x_n^2 x_{n-1} x_2, \quad \dots, \quad u_n = 8x_n^2 x_{n-1} x_{n-2}.$$

And

$$u_1 + u_2 + u_3 + \dots + u_n$$

$$= x_n^4 - 4x_{n-1}x_n^3 + (2s + 8x_{n-1}r - 8x_{n-1}^2)x_n^2 + 4sx_{n-1}x_n + s^2,$$

which is to be a square. Assume the above equal to

$$(x_n^2 + 2x_{n-1}x_n + s)^2 = x_n^4 + 4x_{n-1}x_n^3 + (2s + 4x_{n-1}^2)x_n^2 + 4sx_{n-1}x_n + s^2.$$

From this we find $x_n = r - \frac{2}{3}x_{n-1}$; which completely solves the problem.

To give one or two numerical examples. Suppose $n=3$, and take $x_1=3$, $x_2=\frac{2}{3}$; then $r=3$ and $s=\frac{13}{9}$; whence $x_3=2$; from these we find $u_1 = \frac{211}{9}$, $u_2 = \frac{244}{9}$, $u_3 = \frac{864}{9}$, or multiplying by 81, which we are at liberty to do, we may take $u_1 = 2113$, $u_2 = 744$, $u_3 = 864$.

Again, taking $x_1=4$ and $x_2=2$, we find $r=4$ and $x_3=1$; also $s=20$, which gives $u_1=409$, $u_2=152$, $u_3=64$; and $409^2 + 152^2 + 64^2 = 21^4$.

Again, suppose $n=4$, and take $x_1=5$, $x_2=1$, $x_3=2$; then we find $r=6$, $s=30$; whence $x_4=3$; and we readily find $u_1=1233$, $u_2=504$, $u_3=720$, $u_4=144$, which numbers answer the conditions of the question.

Lastly, suppose $n=5$; and take $x_1=6$, $x_2=2$, $x_3=3$, $x_4=4$; then $r=11$, $s=65$; whence $x_5=5$. From these we find $u_1=4900$, $u_2=3200$, $u_3=4800$, $u_4=1600$, $u_5=2400$. Dividing the above by 100, we may take $x_1=49$, $x_2=32$, $x_3=48$, $x_4=16$, $x_5=24$.

$$\text{Now} \quad 49 + 32 + 48 + 16 + 24 = 169 = 13^2,$$

$$\text{and} \quad 49^2 + 32^2 + 48^2 + 16^2 + 24^2 = 6561 = 81.$$

[The case of $n=2$ is an old problem; of course it is included in the above general solution.]

3898. (Proposed by W. SIVERLY.)—A hollow cone made of thin paper, and having a vertical angle = α , is cut open by a straight line from the vertex to a point in the base, and spread out so as to form a plane surface. Find the vertical angle of this surface.

Solution by the Hon. ALGERNON BOURKE; H. MURPHY; and others.

The surface formed will be a sector of a circle whose radius is the length of slant side of cone; and the length of the arc will be equal to the perimeter of the base of the cone.

Let the slant side = a ; then the perimeter of base = $2\pi a \sin \alpha$; hence the vertical angle of the sector is $2\pi \sin \alpha$.

3938. (Proposed by T. DONSON, B.A.)—If a straight line of a given length has one extremity in a vertical line, and the other in a horizontal plane, each assigned point of the line lies in the surface of a solid. Prove

that the average volume of the solids whose surfaces are the loci of all the points in the given line is two-thirds of the volume of the solid whose surface is the locus of its middle point.

Solution by R. F. SCOTT; A. B. EVANS, M.A.; and many others.

Let c be the length of the whole line; a, b the lengths of the portions into which the assigned point divides it, a being the part next the plane; then the locus of the point is known to be an ellipsoid of revolution, whose axes are a and b, b being the semi-axis of revolution. Hence the volume of this locus is $\frac{4}{3}\pi a^2 b$; and the average volume of all these loci is

$$\frac{4\pi}{3c} \int_0^c a^2 (c-a) da = \frac{1}{3}\pi c^3.$$

This proves the theorem, for the locus of the middle point is a sphere whose diameter is c , and volume $\frac{1}{3}\pi c^3$.

[The *greatest* solid-locus is formed when $a = \frac{2}{3}c, b = \frac{1}{3}c$, its volume being $\frac{8}{27}\pi c^3$; hence the mean solid is $\frac{9}{16}$ of the maximum solid.]

3939. (Proposed by Dr. HIRST, F.R.S.)—Prove that if

$$F(x, y, a, b, c, f, g, h) = 0$$

be the equation of the n th pedal of the conic

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0,$$

the equation of the $(-n+1)$ th pedal will be

$$F\left(-\frac{x}{r^2}, -\frac{y}{r^2}, \frac{d\Delta}{da}, \frac{d\Delta}{db}, \frac{d\Delta}{dc}, \frac{1}{2}\frac{d\Delta}{df}, \frac{1}{2}\frac{d\Delta}{dg}, \frac{1}{2}\frac{d\Delta}{dh}\right) = 0,$$

where $r^2 = x^2 + y^2$, and $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Solution by R. F. SCOTT.

The equation of the reciprocal of the given conic is

$$(A, B, C, F, G, H)(-x, -y, 1)^2 = 0 \dots \dots \dots (1),$$

where

$$A = \frac{d\Delta}{da}, \dots, F = \frac{1}{2}\frac{d\Delta}{df}, \&c.$$

The n th pedal of (1), is $F(-x, -y, A, B, C, F, G, H) = 0 \dots \dots \dots (2).$

But the Proposer has shown (*Reprint*, Vol. I., p. 41) that the inverse of the n th pedal of any curve is the $(-n+1)$ th pedal of the reciprocal of the primitive. Hence

$$F\left(-\frac{x}{r^2}, -\frac{y}{r^2}, A, B, C, F, G, H\right) = 0,$$

which is the inverse of (2), is the $(-n+1)$ th pedal of the given conic

$$(a, b, c, f, g, h)(x, y, 1)^2 = 0.$$

[In Art. 7 of a paper by the Editor on pedal curves, given on p. 24 of Vol. I. of the *Reprint*, it is shown that the second positive focal pedal of an ellipse (or the first pedal of a circle for an excentric point) is a limaçon, whose equation is $r = a(1 + e \cos \theta)$ or $(x^2 + y^2 - aex)^2 - a^2(x^2 + y^2) = 0$. From this, by the aid of Dr. HIRST's theorem, we may find the equation in x and y of the first negative focal pedal, which has been otherwise obtained in the *Reprint*, Vol. XVI., p. 78, and Vol. XVII., p. 92.]

3746. (Proposed by Professor WOLSTENHOLME, M.A.)—A uniform chain is in equilibrium under the action of any forces; from a fixed point O is drawn OP' parallel to the tangent at any point P of the chain, and of length proportional to the tension at P. Prove (1) that the tangent at P' to the locus of P' is parallel to the resultant force at P, and (2) that the ultimate ratio of small corresponding arcs at P' P is proportional to the force at P.

Solution by N'IMPORTE.

Let PQ be an element of the chain; then P'O and OQ' represent tensions at P and Q, and Q'P' represents the resultant of these tensions, but the resultant of the tensions is equal and opposite to the force on PQ, therefore P'Q' represents in magnitude and direction the force on PQ.

Otherwise: let T, N denote the resolved parts of the forces along the tangent and normal to the catenary; then, by the equations of equilibrium,

if t denote the tension at any point P, we have $\frac{dt}{ds} = T$, $\frac{t}{\rho} = N$; and

the locus of P' is given by $r' = ct$, $\theta' = \tan^{-1} \frac{dy}{dx}$;

therefore $\frac{dr'}{ds} = c \frac{dt}{ds} = cT$; and $\frac{d\theta'}{ds} = \frac{dx^2}{ds^2} \cdot \frac{d^2y}{dx^2} = \frac{1}{\rho} = \frac{N}{t} = \frac{cN}{r'}$;

therefore $\frac{r' d\theta'}{dr'} = \frac{N}{S}$, which proves (1);

also $\frac{(r' d\theta')^2 + (dr')^2}{ds^2} = c^2 (N^2 + T^2)$, therefore $\frac{ds'}{ds} = c (N^2 + T^2)^{\frac{1}{2}}$,

which proves (2).

3940. (Proposed by Dr. SALMON, F.R.S.)—Prove that the area of the triangle formed by the polars of the middle points of the sides of a triangle, with respect to any inscribed conic, is equal to the area of the given triangle.

I. *Solution by J. J. WALKER, M.A.*

Taking the given triangle, ABC, as the triangle of reference, any inscribed conic will be of the form

$$l^2\alpha^2 + m^2\beta^2 + n^2\gamma^2 - 2mn\beta\gamma - 2nl\gamma\alpha - 2lm\alpha\beta = 0,$$

and the polars of the middle points of the three sides of the triangle ABC with respect to it are readily found to be

$$l(mc + nb)\alpha + m(nb - mc)\beta + n(mc - nb)\gamma = 0 \quad \dots\dots\dots (1),$$

$$l(na - lc)\alpha + m(na + lc)\beta + n(mc - nb)\gamma = 0 \quad \dots\dots\dots (2),$$

$$l(na - lb)\alpha + m(lb - ma)\beta + n(na + lb)\gamma = 0 \quad \dots\dots\dots (3),$$

and the line at infinity is $a\alpha + b\beta + c\gamma = 0 \quad \dots\dots\dots (4).$

The area (Δ_1) of the triangle contained by these three polars, is

$$\Delta_1 = \frac{\Delta \cdot abc (123)^2}{(423) \times (143) \times (124)},$$

(see Solutions of Question 1733, *Reprint*, Vol. IV., pp. 53—56), where Δ is the area of ABC, (123) is the determinant of the three equations (1) (2) (3), (423) that of (4) (2) (3), (143) that of (1) (4) (3), and (124) that of (1) (2) (4). Now it will be found, on calculating their values, that

$$(123) = 8abc l^2 m^2 n^2, \quad (423) = 4abc l^2 mn,$$

$$(143) = 4abclm^2n, \quad (124) = 4abclmn^2.$$

Hence $\Delta_1 = \Delta$, which proves the theorem.

II. *Solution by W. S. MCCART, M.A.*

The area of a triangle in terms of the areal coordinates of its vertices is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix};$$

the area of the triangle of reference being unity and $x + y + z = 1$.

The equations of the lines joining the middle points of the sides of the triangle of reference are

$$-x + y + z = 0, \quad x - y + z = 0, \quad x + y - z = 0.$$

The areal coordinates of the pole of the first with regard to the inscribed conic $\frac{l}{\lambda} + \frac{m}{\mu} + \frac{n}{\nu} = 0$, are $\frac{m+n}{2l}$, $\frac{l-n}{2l}$, $\frac{l-m}{2l}$; hence the area of the triangle formed by the three poles is

$$\begin{vmatrix} m+n & l-n & l-m \\ m-n & n+l & m-l \\ n-m & n-l & l+m \end{vmatrix} \frac{1}{8lmn};$$

and the value of this is easily seen to be unity, which is also the area of the triangle of reference.

3531. (Proposed by the Rev. W. A. WHITWORTH, M.A.)—If a particle describe an ellipse under the action of a force tending to any fixed point O,

prove that the law of force is $F \propto \frac{DD'^6}{OP \cdot PP'^3}$, where P is the position of the particle, PP' the chord through O, and DD' the diameter parallel to this chord.

I. Solution by Professor WOLSTENHOLME.

Project the ellipse orthogonally into a circle in a plane passing through O; then, if P, Q be any two points on the ellipse, p, q their projections, the area POQ bears a constant ratio to pOq ; hence the circle is described by p under a force tending to O. Also, if F, f be the forces to O in the ellipse and circle, $F = 2lt \frac{QR}{T'}$, $f = 2lt \frac{qr}{T}$, T being the indefinitely small time from P to Q, and QR, qr the subtenses parallel to OP, op ; or $F : f = lQR : qr = DD' : dd'$.

But $f \propto \frac{1}{pO^2 \cdot pp'^3}$, and $pO : PO = dd' : DD' = pp' : PP'$;

therefore, since dd' is constant, $F \propto \frac{DD'^6}{PO^2 \cdot PP'^3}$.

In the particular case when O is the focus S, $PP' \propto PS \cdot SP' \propto DD'^2$, or $F \propto SP^{-2}$. Since $DD'^2 \propto PO \cdot OP'$,

$$F \propto \left(\frac{1}{PO} + \frac{1}{OP'} \right)^{-3} \cdot PO^{-2} \propto PO \left(1 + \frac{PO}{OP'} \right)^{-3},$$

which seems the preferable form of writing the result.

Also if QQ' be the chord of the ellipse which is bisected in D, the time of moving from Q to Q' is $4 \left(\frac{2}{\mu} \right)^{\frac{1}{2}} \cdot \frac{\theta - \sin \theta \cos \theta}{\sin^3 \theta}$, if $\sin \theta$ be the ratio which QQ' bears to the parallel diameter; and the time of describing the whole ellipse is $\frac{4\pi}{\sin^3 \theta} \left(\frac{2}{\mu} \right)^{\frac{1}{2}}$.

II. Solution by G. S. CARR.

Draw the diameters conjugate to CD and CP, meeting PP' in H and E. NEWTON shows, in Prop. 11 of Solution III. of his *Principia*, that $F \propto \frac{PE^3}{OP^3}$, PE becoming equal to AC when PP' passes through the focus.

By the property of the ellipse, $PH \cdot PE = CD^2$,
therefore $PE = \frac{CD^2}{PH} = \frac{DD'^2}{2PP'}$; therefore $F \propto \frac{DD'^6}{OP^2 \cdot PP'^3}$.

3908. (Proposed by Dr. BALL.)—When the effect of an impulsive force upon a free quiescent rigid body is to make the body revolve about an instantaneous axis, prove that the force and the axis are parallel to the principal axes of a section of the momental ellipsoid about the centre of inertia.

I. Solution by Professor TOWNSEND, F.R.S.

The directions, of the initial motion of translation of the centre of gravity, and of the initial axis of rotation round the centre of gravity, being the diameters of the above ellipsoid parallel to the impulsive force and conjugate to the diametral plane containing its line of direction, are therefore in all cases conjugate diameters of the central section of the surface whose plane they determine; hence, when the initial compound motion of the body is equivalent to a pure rotation, the two aforesaid directions being then at right angles, therefore, &c.

II. Solution by Professor WOLSTENHOLME.

Taking the mass of the body as unity, (X, Y, Z) the components of the impulse applied at a point (x, y, z) , $u_1, u_2, u_3, \omega_1, \omega_2, \omega_3$ the component linear and angular velocities instantaneously produced, the axes being the principal axes at the centre of gravity, we have

$u_1 = X, u_2 = Y, u_3 = Z, A\omega_1 = Zy - Yz, B\omega_2 = Xz - Zx, C\omega_3 = Yx - Xy,$
and the condition for a simple solution is

$$\frac{X}{A}(Zy - Yz) + \dots + \dots = 0, \dots \dots \dots (A).$$

Also the directions of the blow and axis are given by the ratios $X : Y : Z ; \omega_1 : \omega_2 : \omega_3$ respectively. Now these directions will be parallel to the principal axes of some section of the momental ellipsoid $Ax^2 + By^2 + Cz^2 = 1$, if they be (1) at right angles to each other, (2) conjugate to each other with respect to the ellipsoid; the conditions for which are respectively

$$X\omega_1 + Y\omega_2 + Z\omega_3 = 0, \text{ which is equivalent to (A),}$$

and $AX\omega_1 + BY\omega_2 + CZ\omega_3 = 0$, or $X(Zy - Yz) + \dots + \dots = 0$, which is identical.

It appears therefore that the direction of the blow and of the initial axis of rotation are (as is well-known) always parallel to two conjugate diameters of the momental ellipsoid; and that when the resulting motion is one of rotation only, these conjugate diameters will be at right angles to each other, and will therefore be the principal axes of the section in which they lie.

3904. (Proposed by Professor WOLSTENHOLME.)—A string lies on a horizontal table in the form of any arc of a cycloid, not including a cusp, and the density at any point varies as the cube of the curvature; impulsive tensions are applied to its ends which are to one another as the respective curvatures: prove that the impulsive tension at any point will be proportional to the curvature, and that the whole string will move without change of form parallel to its axis.

Solution by J. HOPKINSON, D.Sc.

Referring to ROUTH'S *Rigid Dynamics*, Art. 296, let T be the impulsive tension at any point P , $T + dT$ the tension at Q , ϕ the angle which the

tangent at P makes with the tangent at the vertex, u and v the initial velocities of an element parallel to the tangent and normal; then we have

$$u = \mu \rho^2 \frac{dT}{ds}, \text{ and } v = \mu \rho^2 T;$$

but $\frac{du}{ds} = \frac{v}{\rho}$, therefore $\rho^2 \frac{d^2 T}{ds^2} + 3\rho \frac{d\rho}{ds} \frac{dT}{ds} - T = 0$;

but $\rho^2 + s^2 = 16a^2$, hence the solution of the question is

$$(16a^2 - s^2)^{\frac{1}{2}} T = A \sin^{-1} \left(\frac{s}{4a} \right) + B.$$

If the tensions at the extremities are proportional to $(16a^2 - s^2)^{-\frac{1}{2}}$, $A = 0$, and $T = \frac{B}{\rho} = \frac{B}{4a \cos \phi}$; therefore $u = 4\mu a B \sin \phi$, $v = 4\mu a B \cos \phi$;

or each point has the same velocity $4\mu a B$ parallel to the axis, and the whole chain will therefore move parallel to the axis without change of form.

[Professor WOLSTENHOLME remarks that this is a particular case of a general proposition, which is obvious when stated in its general form. If a chain of constant or variable density be laid on a smooth horizontal table in the form of the curve in which it would hang under the action of gravity only, and tangential impulses applied at its ends which are to each other as the finite tensions which would exist there when at rest under the action of gravity, the whole chain will move without change of form in direction of the line which was vertical. The finite tensions and gravity correspond to the impulsive tensions and a constant velocity, and the law of the impulsive tension throughout the chain will be the same as that of the finite tensions when at rest.]

1281. (Proposed by the EDITOR.)—From the middle points of the sides of a triangle draw perpendiculars, proportional in length to these sides, and join the ends of the perpendiculars with the opposite corners of the triangle; then prove (1) that the joining lines will meet in a point, and (2) that the locus of this point is the circumscribed rectangular hyperbola, whose trilinear equation is

$$\frac{\sin(B-C)}{a} + \frac{\sin(C-A)}{b} + \frac{\sin(A-B)}{c} = 0.$$

I. Solution by the PROPOSER.

1. Let θ be the angle which the perpendiculars subtend at the ends of the sides (θ being considered positive when the perpendiculars are inside the triangle, and negative when on the outside), then their lengths will be $\frac{1}{2}a \tan \theta$, $\frac{1}{2}b \tan \theta$, $\frac{1}{2}c \tan \theta$, being thus proportional to the sides; hence the trilinear equations of the joining lines are readily found to be each of the three

$$a \sin(A-\theta) = b \sin(B-\theta) = c \sin(C-\theta);$$

and therefore the joining lines will pass through the point where these equations exist simultaneously.

2. The locus of the point of intersection may be obtained by eliminating

θ from these equations, which can easily be done; and the trilinear equation of the locus is thus found to be the circumscribed conic

$$\beta\gamma \sin(B-C) + \gamma\alpha \sin(C-A) + \alpha\beta \sin(A-B) = 0,$$

which may be put in the form given in the question.

That this conic is a rectangular hyperbola is at once seen from the fact that $\sin(B-C) \cos A + \sin(C-A) \cos B + \sin(A-B) \cos C$ vanishes identically (See WHITWORTH'S *Modern Analytical Geometry*, Art. 247, Cor. 4.)

II. Solution by the Rev. W. A. WHITWORTH, M.A.

1. Taking the given triangle as triangle of reference, the coordinates of the middle point of BC will be 0, $\frac{1}{2}a \sin C$, $\frac{1}{2}a \sin B$. Therefore the line through this point perpendicular to BC, will be represented by

$$\frac{\alpha}{1} = \frac{\beta - \frac{1}{2}a \sin C}{-\cos C} = \frac{\gamma - \frac{1}{2}a \sin B}{-\cos B}$$

(See *Messenger of Mathematics*, Vol. I., pp. 42—44, or WHITWORTH'S *Analytical Geometry*, Art. 68).

Hence if this line be produced to a length ka , its extremity will be given by $\alpha = ka$, $\beta = \frac{1}{2}a \sin C - ka \cos C$, $\gamma = \frac{1}{2}a \sin B - ka \cos B$; and therefore the line joining it with the angular point A is represented by

$$\beta(\sin B - 2k \cos B) = \gamma(\sin C - 2k \cos C).$$

Hence, by symmetry, this line and the other two, drawn similarly with reference to the other two sides, all meet in a point whose equations are

$$\alpha(\cos A - 2k \cos A) = \beta(\sin B - 2k \cos B) = \gamma(\sin C - 2k \cos C).$$

2. Eliminating k , from these equations, the equation to the locus of such points is

$$\begin{vmatrix} \frac{1}{\alpha} & \frac{1}{\beta} & \frac{1}{\gamma} \\ \sin A & \sin B & \sin C \\ \cos A & \cos B & \cos C \end{vmatrix} = 0,$$

$$\text{or} \quad \frac{\sin(B-C)}{\alpha} + \frac{\sin(C-A)}{\beta} + \frac{\sin(A-B)}{\gamma} = 0.$$

